

Integer-Forcing Linear Receivers

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Abstract

Linear receivers are often used to reduce the implementation complexity of multiple antenna systems. In a traditional linear receiver architecture, the receive antennas are used to separate out the codewords sent by each transmit antenna, which can then be decoded individually. Although easy to implement, this approach can be highly sub-optimal when the channel matrix is near singular. This paper develops a new linear receiver architecture that uses the receive antennas to create an effective channel matrix with integer-valued entries. Rather than attempting to recover transmitted codewords directly, the decoder recovers integer combinations of the codewords according to the entries of the effective channel matrix. The codewords are all generated using the same linear code which guarantees that these integer combinations are themselves codewords. If the effective channel is full rank, these integer combinations can then be digitally solved for the original codewords. This paper focuses on the special case where there is no coding across transmit antennas. In this setting, the integer-forcing linear receiver significantly outperforms traditional linear architectures such as the decorrelator and MMSE receiver. In the high SNR regime, the proposed receiver attains the optimal diversity-multiplexing tradeoff for the standard MIMO channel. It is further shown that in an extended MIMO model with interference, the integer-forcing linear receiver achieves the optimal generalized degrees-of-freedom.

Index Terms

MIMO, linear receiver architectures, linear codes, lattice codes

I. INTRODUCTION

It is by now well-known that increasing the number of antennas in a wireless system can significantly increase capacity. Since the seminal papers of Foschini and Gans [1] and Telatar [2], multiple-input multiple-output (MIMO) channels have been thoroughly investigated in theory (see [3] for a survey) and implemented in practice [4]. This capacity gain usually comes at the expense of more complex encoders and decoders and a great deal of work has gone into designing low-complexity MIMO architectures. In this paper, we describe a new low-complexity architecture that can attain significantly higher rates than existing solutions of similar complexity.

We focus on the case where each of the M transmit antennas encodes an independent data stream (see Figure 1). That is, there is no coding across the transmit antennas: each data stream \mathbf{w}_m is encoded separately to form a codeword \mathbf{x}_m of length n . Channel state information is only available to the receiver. From the receiver's perspective, the original data streams are coupled in time through encoding and in space (i.e., across antennas) through the MIMO channel. The joint maximum likelihood (ML) receiver simultaneously performs joint decoding across time and receive antennas. Clearly, this is optimal in terms of both rate and probability of error. However, the computational complexity of jointly processing the data streams is high, and it is difficult to implement this type of receiver in wireless systems when the number of streams is large. Instead, linear receivers such as the decorrelator and minimum-mean-squared error (MMSE) receiver are often used as low-complexity alternatives [5].

J. Zhan, B. Nazer and M. Gastpar were supported in part by NSF grants CCR-0347298, CNS-0627024, and CCF-0830428, and by an Okawa Foundation Research Grant. J. Zhan was supported by an NSF Graduate Fellowship. M. Gastpar was supported by 3TU.CeDICT: Centre for Dependable ICT Systems, The Netherlands, and by a European ERC Starting Grant 259530-ComCom. U. Erez was supported by the U.S. - Israel Binational Science Foundation under grant 2008455 and by the Israel Science Foundation under grant 1557/11. The material in this paper was presented in part at the IEEE International Symposium on Information Theory, Austin, TX, June 2010 and at the 72nd IEEE Vehicular Technology Conference, Ottawa, Canada, September 2010.

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Traditional linear receivers first separate the coupling in space by performing a linear projection at the front-end of the receiver. In order to illustrate this concept and motivate the proposed new approach, we consider the 2×2 MIMO channel characterized by the following matrix:

$$\mathbf{H} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}. \quad (1)$$

The simplest choice of a linear receiver front-end *inverts* the channel matrix. This receiver is usually referred to as the *decorrelator* in the literature. That is, the receiver first applies the matrix

$$\mathbf{H}^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \quad (2)$$

to the received signal. Overall, this converts the original channel into a new equivalent channel characterized by the identity matrix and colored Gaussian noise, i.e., to two scalar channels with correlated noise. The linear receiver then proceeds by separately decoding the output of each of these two channels. The well-known drawback of this approach is that the *noise* vector is also multiplied by the linear receiver front-end matrix given in (2), which alters the variances of its components. In our simple example, if we assume that the original channel had independent additive white Gaussian noise of unit variance, the equivalent noises after the linear receiver front-end have variances of 2 and 5, respectively. That is, while the receiver front-end has nulled out cross-interference, it has also significantly increased the noise levels.

The integer-forcing linear architecture advocated here is based on the recent insight that if on all transmit antennas, the same linear or lattice code is used, then it is possible to not only decode codewords themselves, but also *integer* linear combinations of codewords directly [6]. Let us denote the codeword transmitted on the first antenna by \mathbf{x}_1 and the codeword transmitted on the second antenna by \mathbf{x}_2 . Then, for the simple example matrix from (1), the receiver can decode the integer linear combination $2\mathbf{x}_1 + \mathbf{x}_2$ from the first receive antenna and the combination $\mathbf{x}_1 + \mathbf{x}_2$ from the second receive antenna. From these (following [6]), it is possible to recover linear equations of the data streams over an appropriate finite field, $2\mathbf{w}_1 + \mathbf{w}_2$ and $\mathbf{w}_1 + \mathbf{w}_2$. These equations can in turn be digitally solved for the original data streams. The key point in this example is that the noise variances remain unchanged, which increases the effective SNR per data stream. Note that for more general channel matrices beyond the simple example here, it will also be advantageous to first apply an appropriate linear receiver front-end, albeit following principles very different from merely inverting the channel matrix, as we explain in more detail in the sequel.

In this paper, we first consider the standard MIMO channel and develop a new *integer-forcing* linear receiver architecture that provides multiplexing and diversity gains over traditional linear architectures. Our approach relies on the compute-and-forward framework, which allows linear equations of transmitted messages to be efficiently and reliably decoded over a fading channel [6]. We develop a multiple antenna version of compute-and-forward which employs the antennas at the receiver to rotate the channel matrix towards an effective channel matrix with integer entries. Separate decoders can then recover integer combinations of the transmitted messages, which are finally digitally solved for the original messages. We show that this is much more efficient than using the receive antennas to separate the transmitted codewords and directly decoding each individual codeword. Our analysis uses nested lattice codes originally developed to approach the capacity of point-to-point AWGN and dirty-paper channels [7]–[10] and for which practical implementations were presented in [11] and subsequent works.

Next, we generalize the MIMO channel model to include interference [12], [13] and show that the integer-forcing receiver architecture is an attractive approach to the problem of oblivious interference mitigation. By selecting equation coefficients in a direction that depends on both the interference space and the channel matrix, the proposed architecture reduces the impact of interference and attains a non-trivial gain over traditional linear receivers. Furthermore, we show that the integer-forcing receiver achieves the same generalized degrees of freedom as the joint decoder. Our proof uses techniques from Diophantine approximations, which have also recently been used for interference alignment over fixed channels and the characterization of the degrees of freedom for compute-and-forward [14], [15].

In the remainder of the paper, we start with a formal problem statement in Section II, and then overview the basic existing MIMO receiver architectures and their achievable rates in Section III. In Section IV, we present the integer-forcing receiver architecture and a basic performance analysis. We show that the rate difference between the proposed receiver and traditional linear receivers can be arbitrarily large in Section V. We study the outage

performance of the integer-forcing linear receiver under a slow fading channel model in Section VI. We show that in the case where each antenna encodes an independent data stream, our architecture achieves the same diversity-multiplexing tradeoff as that of the optimal joint decoder. In Section VII, we consider the MIMO channel with interference and show that the integer-forcing receiver can be used to effectively mitigate interference. We characterize the generalized degrees-of-freedom for the integer-forcing receiver and find that it is the same as for the joint decoder.

Throughout the paper, we will use boldface lowercase to refer to vectors, $\mathbf{a} \in \mathbb{R}^M$, and boldface uppercase to refer to matrices, $\mathbf{A} \in \mathbb{R}^{M \times M}$. Let \mathbf{A}^T denote the transpose of a matrix \mathbf{A} and $|\mathbf{A}|$ denote the determinant. Also, let \mathbf{A}^{-1} denote the inverse of \mathbf{A} and $\mathbf{A}^\dagger \triangleq (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T$ denote the pseudoinverse. The notation $\|\mathbf{a}\| \triangleq \sqrt{\sum_i a_i^2}$ will refer to the ℓ_2 -norm of the vector \mathbf{a} while $\|\mathbf{a}\|_\infty \triangleq \max_i |a_i|$ will refer to the ℓ_∞ -norm. Finally, we will use $\lambda_{\text{MAX}}(\mathbf{A})$ and $\lambda_{\text{MIN}}(\mathbf{A})$ to refer to the maximum and minimum singular values of the matrix \mathbf{A} .

II. PROBLEM STATEMENT

The baseband representation of a MIMO channel usually takes values over the complex field. For notational convenience, we will work with the real-valued decomposition of these complex matrices. Recall that any equation of the form $\mathbf{Y} = \mathbf{GX} + \mathbf{Z}$ over the complex field can be represented by its real-valued representation,

$$\begin{bmatrix} \text{Re}(\mathbf{Y}) \\ \text{Im}(\mathbf{Y}) \end{bmatrix} = \begin{bmatrix} \text{Re}(\mathbf{G}) & -\text{Im}(\mathbf{G}) \\ \text{Im}(\mathbf{G}) & \text{Re}(\mathbf{G}) \end{bmatrix} \begin{bmatrix} \text{Re}(\mathbf{X}) \\ \text{Im}(\mathbf{X}) \end{bmatrix} + \begin{bmatrix} \text{Re}(\mathbf{Z}) \\ \text{Im}(\mathbf{Z}) \end{bmatrix}. \quad (3)$$

We will henceforth refer to the $2M \times 2N$ real-valued decomposition of the channel matrix as \mathbf{H} . We will use $2M$ independent encoders and $2M$ independent decoders for the resulting real-valued transmit and receive antennas.¹

Definition 1 (Messages): Each of the $2M$ transmit antennas has a length k data stream (or message) \mathbf{w}_m drawn independently and uniformly from $\mathcal{W} = \{0, 1, 2, \dots, q-1\}^k$.

Definition 2 (Encoders): Each data stream \mathbf{w}_m is mapped onto a length n channel input $\mathbf{x}_m \in \mathbb{R}^{n \times 1}$ by an encoder,

$$\mathcal{E}_m : \mathcal{W} \rightarrow \mathbb{R}^n.$$

An equal power allocation is assumed across transmit antennas

$$\frac{1}{n} \|\mathbf{x}_m\|^2 \leq \text{SNR}.$$

While we formally impose a separate power constraint on each antenna, we note that the performance at high SNR (in terms of the diversity-multiplexing tradeoff) remains unchanged if this is replaced by a sum power constraint over all antennas instead.

Definition 3 (Rate): Each of the $2M$ encoders transmits at the same rate

$$R_{\text{TX}} = \frac{k}{n} \log_2 q.$$

The total rate of the MIMO system is just the number of transmit antennas times the rate, $2MR_{\text{TX}}$.

Remark 1: Since the transmitters do not have knowledge of the channel matrix, we focus on the case where the $2M$ data streams are transmitted at equal rates. We will compare the integer-forcing receiver against successive cancellation V-BLAST schemes with asymmetric rates in Section VI-B.

Definition 4 (Channel): Let $\mathbf{X} \in \mathbb{R}^{2M \times n}$ be the matrix of transmitted vectors,

$$\mathbf{X} = \begin{bmatrix} \mathbf{x}_1^T \\ \vdots \\ \mathbf{x}_{2M}^T \end{bmatrix}. \quad (4)$$

The MIMO channel takes \mathbf{X} as an input, multiplies it by the channel matrix $\mathbf{H} \in \mathbb{R}^{2N \times 2M}$ and adds noise $\mathbf{Z} \in \mathbb{R}^{2N \times n}$ whose entries are i.i.d. Gaussian with zero mean and unit variance. The signal $\mathbf{Y} \in \mathbb{R}^{2N \times n}$ observed across the $2N$ receive antennas over n channel uses can be written as

$$\mathbf{Y} = \mathbf{HX} + \mathbf{Z}. \quad (5)$$

¹The implementation complexity of our scheme can be decreased slightly by specializing it to the complex field using the techniques in [6]. For notational convenience, we focus solely on the real-valued representation, and do not exploit the constraints on the matrix \mathbf{H} .

We assume that the channel realization \mathbf{H} is known to the receiver but unknown to the transmitter and remains constant throughout the transmission block of length n .

Definition 5 (Decoder): At the receiver, a *decoder* makes an estimate of the messages,

$$\mathcal{D} : \mathbb{R}^{2N \times n} \rightarrow \mathcal{W}^{2M} \quad (6)$$

$$(\hat{\mathbf{w}}_1, \dots, \hat{\mathbf{w}}_{2M}) = \mathcal{D}(\mathbf{y}). \quad (7)$$

Definition 6 (Achievable Rates): We say that sum rate $R(\mathbf{H})$ is *achievable* if for any $\epsilon > 0$ and n large enough, there exist encoders and a decoder such that reliable decoding is possible

$$\Pr((\hat{\mathbf{w}}_1, \dots, \hat{\mathbf{w}}_{2M}) \neq (\mathbf{w}_1, \dots, \mathbf{w}_{2M})) \leq \epsilon$$

so long as the total rate does not exceed $R(\mathbf{H})$,

$$2MR_{\text{TX}} \leq R(\mathbf{H}).$$

III. EXISTING RECEIVER ARCHITECTURES

Many approaches to MIMO decoding have been studied in the literature. We provide a brief summary of some of the major receiver architectures and the associated achievable rates, including the joint ML receiver, the decorrelator, linear MMSE estimator and the MMSE-SIC estimator.

A. Joint ML Receivers

Clearly, the best performance is attainable by joint ML decoding across all N receive antennas. This situation is illustrated in Figure 1. Let $\mathbf{H}_{\mathcal{S}}$ denote the submatrix of \mathbf{H} formed by taking the columns with indices in $\mathcal{S} \subseteq \{1, 2, \dots, 2M\}$. If we use a joint ML decoder that searches for the most likely set of transmitted messages vectors $\hat{\mathbf{w}}_1, \dots, \hat{\mathbf{w}}_{2M}$, then the following rate is achievable (using Gaussian codebooks at the transmitter):

$$R_{\text{JOINT}}(\mathbf{H}) = \min_{\mathcal{S} \subseteq \{1, 2, \dots, 2M\}} \frac{M}{|\mathcal{S}|} \log \det (\mathbf{I}_{\mathcal{S}} + \text{SNR } \mathbf{H}_{\mathcal{S}} \mathbf{H}_{\mathcal{S}}^T) \quad (8)$$

where \mathbf{I} is the identity matrix.² Note that this is also the capacity of the channel subject to equal rate constraints per transmit antenna. The worst-case complexity of this approach is exponential in the product of the blocklength n and the number of antennas N .

One approach to reduce the complexity of the joint ML decoder is to employ a sphere decoder. Rather than naively checking all possible codewords, the sphere decoder only examines codewords that lie within a ball around the received vector. If the radius of the ball is suitably chosen, this search is guaranteed to return the ML candidate vector. We refer interested readers to [16]–[20] for more details on sphere decoding algorithms as well as to [21] for a recent hardware implementation.

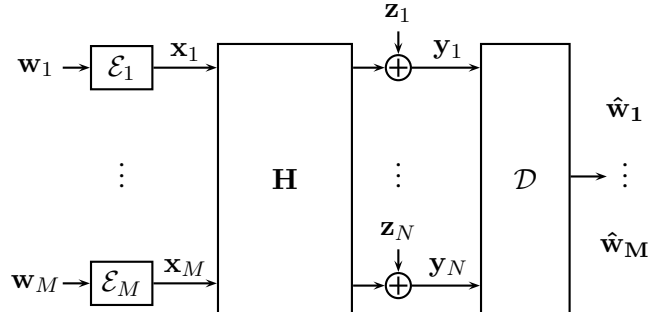


Fig. 1. MIMO channel with single stream encoding.

²With joint encoding and decoding, a rate of $\frac{1}{2} \log \det (\mathbf{I} + \text{SNR} \mathbf{H} \mathbf{H}^T)$ is achievable.

B. Traditional Linear Receivers

Rather than processing all the observed signals from the antennas jointly, one simple approach is to separate out the transmitted data streams using a linear projection and then decode each data stream individually, as shown in Figure 2. Given the observed matrix $\mathbf{Y} = \mathbf{H}\mathbf{X} + \mathbf{Z}$ from (5), the receiver forms the projection

$$\tilde{\mathbf{Y}} = \mathbf{B}\mathbf{Y} \quad (9)$$

$$= \mathbf{B}\mathbf{H}\mathbf{X} + \mathbf{B}\mathbf{Z} \quad (10)$$

where $\mathbf{B} \in \mathbb{R}^{2M \times 2N}$. Each row $\tilde{\mathbf{y}}_m^T$ of $\tilde{\mathbf{Y}}$ is treated as a noisy version of \mathbf{x}_m^T . In traditional linear receivers, the goal of the projection matrix \mathbf{B} is to separate the incoming data streams. For the decorrelator architecture, we choose the projection to be the pseudoinverse of the channel matrix $\mathbf{B} = (\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T$. In the case where $N \geq M$, the resulting channel is interference free. If \mathbf{H} is orthogonal, then the decorrelator architecture can match the performance of a joint ML decoder. As the condition number of \mathbf{H} increases, the performance gap between the decorrelator and the joint decoder increases due to noise amplification (see the example in Section V-B). The performance of the decorrelator can be improved at low SNR using the MMSE architecture which sets $\mathbf{B} = \mathbf{H}^T(\mathbf{H}\mathbf{H}^T + \frac{1}{\text{SNR}}\mathbf{I})^{-1}$. Let \mathbf{b}_m^T be the m^{th} row vector of \mathbf{B} and \mathbf{h}_m the m^{th} column vector of \mathbf{H} . The following rate is achievable for the m^{th} data stream using a decorrelator architecture with Gaussian codebooks:

$$R_m(\mathbf{H}) = \frac{1}{2} \log \left(1 + \frac{\text{SNR} \|\mathbf{b}_m^T \mathbf{h}_m\|^2}{\|\mathbf{b}_m\|^2 + \text{SNR} \sum_{i \neq m} \|\mathbf{b}_m^T \mathbf{h}_i\|^2} \right). \quad (11)$$

Since we focus on the case where each data stream is encoded at the same rate, the achievable sum rate is dictated by the worst stream,

$$R_{\text{LINEAR}}(\mathbf{H}) = \min_m 2MR_m(\mathbf{H}). \quad (12)$$

The complexity of a linear receiver architecture is dictated primarily by the choice of decoding algorithm for the individual data streams. In the worst case (when ML decoding is used for each data stream), the complexity is exponential in the blocklength of the data stream. In practice, one can employ low-density parity-check (LDPC) codes to approach rates close to the capacity with linear complexity [22].

The performance of this class of linear receivers can be improved using successive interference cancellation (SIC) [23], [24]. After a codeword is decoded, it may be subtracted from the observed vector prior to decoding the next codeword, which increases the effective signal-to-noise ratio. Let Π denote the set of all permutations of $\{1, 2, \dots, 2M\}$. For a fixed decoding order $\pi \in \Pi$, let $\pi_m = \{\pi(m), \pi(m+1), \dots, \pi(2M)\}$ denote the indices of the data streams that have not yet been decoded. Let $\mathbf{h}_{\pi(m)}$ denote the $\pi(m)^{\text{th}}$ column vector of \mathbf{H} and let \mathbf{H}_{π_m} be the submatrix consisting of the columns with indices π_m , i.e., $\mathbf{H}_{\pi_m} = [\mathbf{h}_{\pi(m)} \cdots \mathbf{h}_{\pi(2M)}]$. The following rate is achievable in the $\pi(m)^{\text{th}}$ stream using successive interference cancellation:

$$R_{\pi(m)}(\mathbf{H}) = \frac{1}{2} \log \left(1 + \frac{\text{SNR} \|\mathbf{b}_m^T \mathbf{h}_{\pi(m)}\|^2}{\|\mathbf{b}_m\|^2 + \text{SNR} \sum_{i > m} \|\mathbf{b}_m^T \mathbf{h}_{\pi(i)}\|^2} \right). \quad (13)$$

where $\mathbf{b}_m = (\mathbf{H}_{\pi_m} \mathbf{H}_{\pi_m}^T + \frac{1}{\text{SNR}}\mathbf{I})^{-1} \mathbf{h}_{\pi(m)}$ is the projection vector to decode the $\pi(m)^{\text{th}}$ stream after canceling the interference from the $\pi(1), \dots, \pi(m-1)^{\text{th}}$ streams. For a fixed decoding order π , the achievable sum-rate is given by

$$R_{\text{SIC},1}(\mathbf{H}) = \min_m 2MR_{\pi(m)}(\mathbf{H}). \quad (14)$$

The above scheme is referred to as V-BLAST I (see [25] for more details). An improvement can be attained by selecting the decoding order, and thus the permutation.

In the case of V-BLAST II, the sum rate is improved by choosing the decoding order that maximizes rate of the worst stream,

$$R_{\text{SIC},2}(\mathbf{H}) = \max_{\pi \in \Pi} \min_m 2MR_{\pi(m)}(\mathbf{H}). \quad (15)$$

Hence, V-BLAST I performs worse than V-BLAST II for all channel parameters. We postpone the discussion of V-BLAST III to Section VI where we introduce the outage formulation.

Using ML decoding for each individual data stream, the complexity of the MMSE-SIC architecture is again exponential in blocklength. However, unlike the decorrelator and linear MMSE receiver, not all M streams can be decoded in parallel and delay is incurred as later streams have to wait for earlier streams to finish decoding.

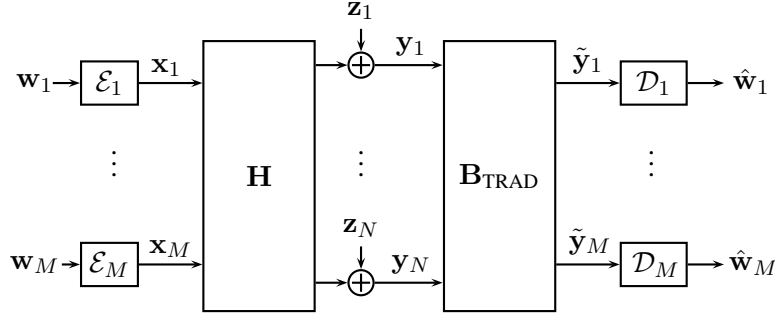


Fig. 2. A traditional linear receiver. Each of the individual message vectors is decoded directly from the projected channel output. The goal of the linear projection is to approximately invert the channel and cancel the interference from other streams.

C. Lattice-Reduction Detectors

Another class of linear architectures comes under the name of *lattice-reduction* detectors. It has been shown that lattice reduction can be used to improve the performance of the decorrelator when the channel matrix is near singular [26] and can achieve the receive diversity [27]. Lattice-reduction detectors are symbol-level linear receivers that impose a linear constellation constraint, *e.g.*, a QAM constellation, on the transmitters. The output of the MIMO channel \mathbf{Y} is passed through a linear filter \mathbf{B} to get the resulting output:

$$\tilde{\mathbf{Y}} = \mathbf{B}\mathbf{Y} \quad (16)$$

$$= \mathbf{B}\mathbf{H}\mathbf{X} + \mathbf{B}\mathbf{Z} \quad (17)$$

$$= \mathbf{A}\mathbf{X} + \mathbf{B}\mathbf{Z} \quad (18)$$

where $\mathbf{A} = \mathbf{B}\mathbf{H}$ is the effective channel matrix. In lattice reduction, the effective channel matrix is restricted to be unimodular: both its entries and the entries of its inverse must be integers. Let $\mathbf{a}_1^T, \dots, \mathbf{a}_{2M}^T$ be the row vectors of matrix \mathbf{A} . The lattice-reduction detector produces estimates of the symbols of $\mathbf{a}_m^T \mathbf{X}$ from $\tilde{\mathbf{Y}}$. There are two key differences between the proposed integer-forcing receiver and the lattice-reduction receiver. First, the integer-forcing receiver operates on the codeword level rather than on the symbol level. Second, the effective channel matrix \mathbf{A} of the integer-forcing receiver is not restricted to be unimodular: it can be any full-rank integer matrix. We compare lattice reduction to the integer receiver in Example 3 by restricting the effective matrix to be unimodular under the integer-forcing architecture. We show that this restriction can result in an arbitrarily large performance gap.

Two other works have developed lattice architectures for joint decoding that can achieve the optimal diversity-multiplexing tradeoff [28], [29].

IV. PROPOSED RECEIVER ARCHITECTURE

A. Architecture Overview

Linear receivers such as the decorrelator and the MMSE receiver directly decode the data streams after the projection step. In other words, they use the linear projection matrix \mathbf{B} to invert the channel matrix at the cost of amplifying the noise. Although low in complexity, these approaches are far from optimal when the channel matrix is ill-conditioned. In the integer-forcing architecture, each encoder uses the same linear code and the receiver exploits the code-level linearity to recover equations of the transmitted messages. Instead of inverting the channel, the scheme uses \mathbf{B} to force the effective channel to a full-rank integer matrix \mathbf{A} . As in the case of traditional linear receivers, each element of the effective output is then sent to a separate decoder. However, since each encoder uses

the same linear code, each decoder can recover an integer linear combination of the codewords. The integer-forcing receiver is free to choose the set of equation coefficients \mathbf{A} to be any full-rank integer matrix. The resulting integer combinations of codewords can be mapped back to a set of full-rank messages over a finite field.³ Finally, the individual messages vectors are recovered from the set of full-rank equations of message vectors. The details of the architecture are provided in the sequel and an illustration is given in Figure 3.

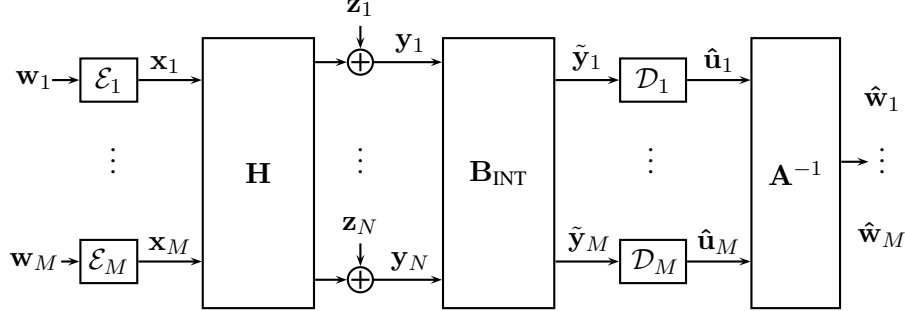


Fig. 3. The proposed integer-forcing linear receiver. Each decoder first recovers an equation of the transmitted message vectors. These equations are then collected and solved for the individual message vectors. The goal of the linear projection is to create a full-rank, integer-valued effective channel matrix.

Prior to decoding, our receiver projects the channel output using the $2M \times 2N$ matrix \mathbf{B} to get the effective channel

$$\tilde{\mathbf{Y}} = \mathbf{B}\mathbf{Y} \quad (19)$$

$$= \mathbf{B}\mathbf{H}\mathbf{X} + \mathbf{B}\mathbf{Z}. \quad (20)$$

Each preprocessed output $\tilde{\mathbf{y}}_m$ is then passed into a separate decoder $\mathcal{D}_m : \mathbb{R}^n \rightarrow \mathcal{W}$. Decoder m attempts to recover a linear equation of the message vectors

$$\mathbf{u}_m = \left[\sum_{\ell=1}^{2M} a_{m,\ell} \mathbf{w}_\ell \right] \bmod q \quad (21)$$

for some $a_{m,\ell} \in \mathbb{Z}$. Let \mathbf{a}_m denote the vector of desired coefficients for decoder m , $\mathbf{a}_m = [a_{m1} \ a_{m2} \ \cdots \ a_{m2M}]^T$. We choose $\mathbf{a}_1, \dots, \mathbf{a}_{2M}$ to be linearly independent.⁴ Decoder m outputs an estimate $\hat{\mathbf{u}}_m$ for the equation \mathbf{u}_m . We will design our scheme such that, for any $\epsilon > 0$ and n large enough, the desired linear equations are recovered with probability of error satisfying

$$\Pr((\hat{\mathbf{u}}_1, \dots, \hat{\mathbf{u}}_{2M}) \neq (\mathbf{u}_1, \dots, \mathbf{u}_{2M})) \leq \epsilon. \quad (22)$$

Let $\mathbf{W} = [\mathbf{w}_1 \ \cdots \ \mathbf{w}_{2M}]^T$ denote the matrix of message vectors, $\mathbf{U} = [\mathbf{u}_1 \ \cdots \ \mathbf{u}_{2M}]^T$ denote the matrix of linear equations of message vectors and $\mathbf{A} = [\mathbf{a}_1 \ \cdots \ \mathbf{a}_{2M}]^T$ denote the integer matrix of equation coefficients. Since \mathbf{A} is full-rank, the original message vectors can be recovered from the set of linear equations by a simple inverse operation:

$$\mathbf{W} = \mathbf{A}^{-1}\mathbf{U} \quad (23)$$

In the following subsections, we will provide details on the achievable rate, the choice of the coefficients of the integer matrix \mathbf{A} , and the complexity of our architecture.

³For the scope of the present paper, we assume that q is prime to ensure invertibility. However, this restriction may be removed as shown in [30].

⁴It is sufficient to consider matrices \mathbf{B} and desired coefficient vectors \mathbf{a}_m that are real-valued decompositions of a complex matrix or vector.

B. Achievable Rates

We use the compute-and-forward framework developed in [6] to derive the achievable rate of the integer-forcing linear receiver. Let \mathbf{h}_m^T be the m^{th} row vector of \mathbf{H} . In the case where $\mathbf{B} = \mathbf{I}$, the channel output to the m^{th} decoder is given by

$$\mathbf{y}_m^T = \mathbf{h}_m^T \mathbf{X} + \mathbf{z}_m^T \quad (24)$$

and the rate at which the set of equations $\mathbf{u}_1, \dots, \mathbf{u}_{2M}$ can be reliably recovered is given in the following theorem. Define $\log^+(x) \triangleq \max\{x, 0\}$.

Theorem 1 ([6, Theorem 1]): For any $\epsilon > 0$ and n large enough, there exist fixed encoders and decoders, $\mathcal{E}_1, \dots, \mathcal{E}_{2M}, \mathcal{D}_1, \dots, \mathcal{D}_{2M}$, such that all decoders can recover their equations with total probability of error at most ϵ so long as

$$R_{TX} < \min_{m=1, \dots, 2M} R(\mathbf{H}, \mathbf{a}_m) \quad (25)$$

$$R(\mathbf{H}, \mathbf{a}_m) = \frac{1}{2} \log^+ \left(\frac{\text{SNR}}{1 + \text{SNR} \|\mathbf{h}_m - \mathbf{a}_m\|^2} \right) \quad (26)$$

for the selected equation coefficients $\mathbf{a}_1, \dots, \mathbf{a}_{2M} \in \mathbb{Z}^{2M}$.

Remark 2: Note that the decoders in Theorem 1 are free to choose any equation coefficients that satisfy (25). The encoders are completely oblivious to the choice of coefficients.

It is instructive to examine the noise term $1 + \text{SNR} \|\mathbf{h}_m - \mathbf{a}_m\|^2$. The leading 1 corresponds to the additive noise, which has unit variance in our model. The more interesting term $\|\mathbf{h}_m - \mathbf{a}_m\|^2$ corresponds to the “non-integer” penalty since the channel coefficients \mathbf{h}_m are not exactly matched to the coefficients \mathbf{a}_m of the linear equation.

As illustrated in Figure 3, we first multiply the channel output matrix \mathbf{Y} by a judiciously chosen matrix \mathbf{B} . That is, the effective channel output observed by the m^{th} decoder can be expressed as

$$\tilde{\mathbf{y}}_m^T = \sum_{i=1}^{2M} (\mathbf{b}_m^T \mathbf{h}_i) \mathbf{x}_i^T + \mathbf{b}_m^T \mathbf{Z} \quad (27)$$

$$= \tilde{\mathbf{h}}_m^T \mathbf{X} + \tilde{\mathbf{z}}_m^T \quad (28)$$

where $\tilde{\mathbf{h}}_m = \mathbf{H}^T \mathbf{b}_m$ is the effective channel to the m^{th} decoder and $\tilde{\mathbf{z}}_m$ is the effective noise with variance $\|\mathbf{b}_m\|^2$. The achievable rate of the integer-forcing linear receiver is given in the next theorem.

Theorem 2: Consider the MIMO channel with channel matrix $\mathbf{H} \in \mathbb{R}^{2N \times 2M}$. Under the integer-forcing architecture, the following rate is achievable:

$$R < \min_m 2M R(\mathbf{H}, \mathbf{a}_m, \mathbf{b}_m) \quad (29)$$

$$R(\mathbf{H}, \mathbf{a}_m, \mathbf{b}_m) = \frac{1}{2} \log^+ \left(\frac{\text{SNR}}{\|\mathbf{b}_m\|^2 + \text{SNR} \|\mathbf{H}^T \mathbf{b}_m - \mathbf{a}_m\|^2} \right)$$

for any full-rank integer matrix $\mathbf{A} \in \mathbb{Z}^{2M \times 2M}$ and any matrix $\mathbf{B} \in \mathbb{R}^{2M \times 2N}$.

Proof: Applying Theorem 1 with effective channel $\tilde{\mathbf{h}}_m = \mathbf{H}^T \mathbf{b}_m$ and effective noise variance $\|\mathbf{b}_m\|^2$, it follows that the receiver can reliably recover the set of linear equations $\mathbf{u}_1, \dots, \mathbf{u}_{2M}$ where

$$\mathbf{u}_m = \left[\sum_{\ell=1}^{2M} a_{m,\ell} \mathbf{w}_\ell \right] \bmod q. \quad (30)$$

The message vectors $\mathbf{w}_1, \dots, \mathbf{w}_{2M}$ can be solved in turn by inverting the linear equations, $\mathbf{W} = \mathbf{A}^{-1} \mathbf{U}$. ■

Theorem 2 provides an achievable rate for the integer-forcing architecture for any preprocessing matrix \mathbf{B} and any full-rank integer matrix \mathbf{A} . The remaining task is to select these matrices in such a way as to *maximize* the rate expression given in Theorem 2. This turns out to be a non-trivial task. We consider it in two steps. In particular, we first observe that for a fixed integer matrix \mathbf{A} , it is straightforward to characterize the optimal preprocessing matrix \mathbf{B} . Then, in the next subsection, we discuss the problem of selecting the integer matrix \mathbf{A} .

We consider the case when $N \geq M$ and note that given a fixed full-rank integer matrix \mathbf{A} , a simple choice for preprocessing matrix is

$$\mathbf{B}_{\text{EXACT}} = \mathbf{H}^\dagger \mathbf{A} \quad (31)$$

where \mathbf{H}^\dagger is the pseudoinverse of \mathbf{H} . We call this scheme “exact” integer-forcing since the effective channel matrix after preprocessing is simply the full-rank integer matrix \mathbf{A} . We also note that choosing $\mathbf{B}_{\text{EXACT}}$ and $\mathbf{A} = \mathbf{I}$ corresponds to the decorrelator. More generally, the performance of exact integer-forcing is summarized in the following corollary.

Corollary 1: Consider the case where $N \geq M$. The achievable rate from Theorem 2 can be written equivalently as

$$R < \min_m 2MR(\mathbf{H}, \mathbf{a}_m) \quad (32)$$

$$R(\mathbf{H}, \mathbf{a}_m) = \frac{1}{2} \log \left(\frac{\text{SNR}}{\|(\mathbf{H}^T)^\dagger \mathbf{a}_m\|^2} \right) \quad (33)$$

for any full-rank integer matrix \mathbf{A} by setting $\mathbf{B} = \mathbf{H}^\dagger \mathbf{A}$.

We call the expression in the denominator of (33) the “effective noise variance.” The achievable rate in (32) is determined by the largest effective noise variance,

$$\tilde{\sigma}_{\text{EFFECTIVE}}^2 = \max_m \|(\mathbf{H}^T)^\dagger \mathbf{a}_m\|^2. \quad (34)$$

Hence, the goal is to choose linearly independent equations $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_{2M}$ to minimize the expression $\tilde{\sigma}_{\text{EFFECTIVE}}^2$ in (34). The integer-forcing receiver provides the freedom to choose any full-rank integer matrix \mathbf{A} . In the remainder of this section, we characterize the optimal linear projection matrix \mathbf{B} for a fixed coefficient matrix \mathbf{A} and provide an equivalent rate expression for Theorem 2. We will then use this expression in the Section IV-C to provide insight into how to select the optimal coefficient matrix \mathbf{A} .

Corollary 2: The optimal linear projection matrix for a fixed coefficient matrix \mathbf{A} is given by

$$\mathbf{B}_{\text{OPT}} = \mathbf{A} \mathbf{H}^T \left(\frac{1}{\text{SNR}} \mathbf{I} + \mathbf{H} \mathbf{H}^T \right)^{-1}. \quad (35)$$

Remark 3: The linear MMSE estimator, given by $\mathbf{B}_{\text{MMSE}} = \mathbf{H}^T (\frac{1}{\text{SNR}} \mathbf{I} + \mathbf{H} \mathbf{H}^T)^{-1}$, is a special case of the integer-forcing receiver with \mathbf{B}_{OPT} and $\mathbf{A} = \mathbf{I}$.

Remark 4: $\lim_{\text{SNR} \rightarrow \infty} \mathbf{B}_{\text{OPT}} = \mathbf{B}_{\text{EXACT}}$. Hence, under a fixed channel matrix, exact integer-forcing is optimal as $\text{SNR} \rightarrow \infty$.

Proof of Corollary 2: Let $\mathbf{B} = [\mathbf{b}_1, \dots, \mathbf{b}_{2M}]^T$. We solve for each \mathbf{b}_m separately to maximize the achievable rate in Theorem 2,

$$\begin{aligned} \mathbf{b}_m &= \arg \max_{\mathbf{b}_m} \frac{1}{2} \log \left(\frac{\text{SNR}}{\|\mathbf{b}_m\|^2 + \text{SNR} \|\mathbf{H}^T \mathbf{b}_m - \mathbf{a}_m\|^2} \right) \\ &= \arg \min_{\mathbf{b}_m} \frac{1}{\text{SNR}} \|\mathbf{b}_m\|^2 + \|\mathbf{H}^T \mathbf{b}_m - \mathbf{a}_m\|^2. \end{aligned} \quad (36)$$

Define this quantity to be the function $f(\mathbf{b}_m)$ and rewrite as follows:

$$f(\mathbf{b}_m) = \frac{1}{\text{SNR}} \|\mathbf{b}_m\|^2 + \|\mathbf{H}^T \mathbf{b}_m - \mathbf{a}_m\|^2 \quad (37)$$

$$= \frac{1}{\text{SNR}} \mathbf{b}_m^T \mathbf{b}_m + (\mathbf{H}^T \mathbf{b}_m - \mathbf{a}_m)^T (\mathbf{H}^T \mathbf{b}_m - \mathbf{a}_m) \quad (38)$$

$$= \frac{1}{\text{SNR}} \mathbf{b}_m^T \mathbf{b}_m + \mathbf{b}_m^T \mathbf{H} \mathbf{H}^T \mathbf{b}_m - 2 \mathbf{b}_m^T \mathbf{H} \mathbf{a}_m + \mathbf{a}_m^T \mathbf{a}_m \quad (39)$$

$$= \mathbf{b}_m^T \left(\frac{1}{\text{SNR}} \mathbf{I} + \mathbf{H} \mathbf{H}^T \right) \mathbf{b}_m - 2 \mathbf{b}_m^T \mathbf{H} \mathbf{a}_m + \mathbf{a}_m^T \mathbf{a}_m \quad (40)$$

$$(41)$$

Taking the derivative of f with respect to \mathbf{b}_m , we have that

$$\frac{df(\mathbf{b}_m)}{d\mathbf{b}_m} = 2 \left(\frac{1}{\text{SNR}} \mathbf{I} + \mathbf{H}\mathbf{H}^T \right) \mathbf{b}_m - 2\mathbf{H}\mathbf{a}_m. \quad (42)$$

Setting $\frac{df(\mathbf{b}_m)}{d\mathbf{b}_m} = 0$ and solving for \mathbf{b}_m , we have that

$$\mathbf{b}_m^T = \mathbf{a}_m^T \mathbf{H}^T \left(\frac{1}{\text{SNR}} \mathbf{I} + \mathbf{H}\mathbf{H}^T \right)^{-1}. \quad (43)$$

Corollary 2 follows since $\mathbf{B} = [\mathbf{b}_1, \dots, \mathbf{b}_{2M}]^T$. ■

Using the optimal linear projection matrix from Corollary 2, we derive an alternative expression for the achievable rate in Theorem 2.

Theorem 3: The achievable rate from Theorem 2 under the optimal projection matrix \mathbf{B}_{OPT} from (35) can be expressed as

$$R < \min_m 2MR(\mathbf{H}, \mathbf{a}_m) \quad (44)$$

$$R(\mathbf{H}, \mathbf{a}_m) = -\frac{1}{2} \log \mathbf{a}_m^T \mathbf{V} \mathbf{D} \mathbf{V}^T \mathbf{a}_m, \quad (45)$$

where $\mathbf{V} \in \mathbb{R}^{2M \times 2M}$ is the matrix composed of the eigenvectors of $\mathbf{H}^T \mathbf{H}$ and $\mathbf{D} \in \mathbb{R}^{2M \times 2M}$ is a diagonal matrix with elements

$$\mathbf{D}_{i,i} = \begin{cases} \frac{1}{\text{SNR}\lambda_i^2 + 1} & i \leq \text{rank}(\mathbf{H}) \\ 1 & i > \text{rank}(\mathbf{H}) \end{cases} \quad (46)$$

and λ_i is the i^{th} singular value of \mathbf{H} .

Proof: Let f be defined as in (37) and define $\mathbf{U}\mathbf{\Sigma}\mathbf{V}^T$ to be the singular value decomposition (SVD) of \mathbf{H} with $\mathbf{U} \in \mathbb{R}^{2N \times 2N}$, $\mathbf{\Sigma} \in \mathbb{R}^{2N \times 2M}$, and $\mathbf{V} \in \mathbb{R}^{2M \times 2M}$. Note that in this SVD representation, $\Sigma_{i,i} = \lambda_i$ and $\Sigma_{i,j} = 0$ for all $i \neq j$. Evaluating f for the m^{th} row \mathbf{b}_m of \mathbf{B}_{OPT} yields

$$f(\mathbf{b}_m) = \frac{1}{\text{SNR}} \mathbf{b}_m^T \mathbf{b}_m + \mathbf{b}_m^T \mathbf{H}\mathbf{H}^T \mathbf{b}_m - \mathbf{b}_m^T \mathbf{H}\mathbf{a}_m - \mathbf{a}_m^T \mathbf{H}^T \mathbf{b}_m + \mathbf{a}_m^T \mathbf{a}_m \quad (47)$$

$$= \mathbf{b}_m^T \left(\frac{1}{\text{SNR}} \mathbf{I} + \mathbf{H}\mathbf{H}^T \right) \mathbf{b}_m - \mathbf{b}_m^T \mathbf{H}\mathbf{a}_m - \mathbf{a}_m^T \mathbf{H}^T \mathbf{b}_m + \mathbf{a}_m^T \mathbf{a}_m \quad (48)$$

Combining (43) and (48), it follows that

$$f(\mathbf{b}_m) = \mathbf{b}_m^T \left(\frac{1}{\text{SNR}} \mathbf{I} + \mathbf{H}\mathbf{H}^T \right) \left(\frac{1}{\text{SNR}} \mathbf{I} + \mathbf{H}\mathbf{H}^T \right)^{-1} \mathbf{H}\mathbf{a}_m - \mathbf{b}_m^T \mathbf{H}\mathbf{a}_m - \mathbf{a}_m^T \mathbf{H}^T \mathbf{b}_m + \mathbf{a}_m^T \mathbf{a}_m \quad (49)$$

$$= \mathbf{b}_m^T \mathbf{H}\mathbf{a}_m - \mathbf{b}_m^T \mathbf{H}\mathbf{a}_m - \mathbf{a}_m^T \mathbf{H}^T \mathbf{b}_m + \mathbf{a}_m^T \mathbf{a}_m \quad (50)$$

$$= -\mathbf{a}_m^T \mathbf{H}^T \mathbf{b}_m + \mathbf{a}_m^T \mathbf{a}_m \quad (51)$$

$$= -\mathbf{a}_m^T \mathbf{H}^T \left(\frac{1}{\text{SNR}} \mathbf{I} + \mathbf{H}\mathbf{H}^T \right)^{-1} \mathbf{H}\mathbf{a}_m + \mathbf{a}_m^T \mathbf{a}_m \quad (52)$$

$$= -\mathbf{a}_m^T \mathbf{V}\mathbf{\Sigma}^T \mathbf{U}^T \left(\frac{1}{\text{SNR}} \mathbf{I} + \mathbf{U}\mathbf{\Sigma}\mathbf{\Sigma}^T \mathbf{U}^T \right)^{-1} \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T \mathbf{a}_m + \mathbf{a}_m^T \mathbf{I} \mathbf{a}_m. \quad (53)$$

Since \mathbf{U} is an orthonormal matrix, $\mathbf{U}^{-1} = \mathbf{U}^T$ and (53) can be rewritten as follows

$$f(\mathbf{b}_m) = -\mathbf{a}_m^T \mathbf{V}\mathbf{\Sigma}^T \mathbf{U}^T \left(\frac{1}{\text{SNR}} \mathbf{U}\mathbf{U}^T + \mathbf{U}\mathbf{\Sigma}\mathbf{\Sigma}^T \mathbf{U}^T \right)^{-1} \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T \mathbf{a}_m + \mathbf{a}_m^T \mathbf{I} \mathbf{a}_m \quad (54)$$

$$= -\mathbf{a}_m^T \mathbf{V}\mathbf{\Sigma}^T \mathbf{U}^T (\mathbf{U}^T)^{-1} \left(\frac{1}{\text{SNR}} \mathbf{I} + \mathbf{\Sigma}\mathbf{\Sigma}^T \right)^{-1} \mathbf{U}^{-1} \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T \mathbf{a}_m + \mathbf{a}_m^T \mathbf{I} \mathbf{a}_m \quad (55)$$

$$= -\mathbf{a}_m^T \mathbf{V}\mathbf{\Sigma}^T \left(\frac{1}{\text{SNR}} \mathbf{I} + \mathbf{\Sigma}\mathbf{\Sigma}^T \right)^{-1} \mathbf{\Sigma}\mathbf{V}^T \mathbf{a}_m + \mathbf{a}_m^T \mathbf{I} \mathbf{a}_m. \quad (56)$$

Since \mathbf{V} is an orthonormal matrix, $\mathbf{V}^{-1} = \mathbf{V}^T$ and (56) can be rewritten as follows

$$f(\mathbf{b}_m) = \mathbf{a}_m^T \left(-\mathbf{V} \Sigma^T \left(\frac{1}{\text{SNR}} \mathbf{I} + \Sigma \Sigma^T \right)^{-1} \Sigma \mathbf{V}^T + \mathbf{V} \mathbf{V}^T \right) \mathbf{a}_m \quad (57)$$

$$= \mathbf{a}_m^T \mathbf{V} \left(-\Sigma^T \left(\frac{1}{\text{SNR}} \mathbf{I} + \Sigma \Sigma^T \right)^{-1} \Sigma + \mathbf{I} \right) \mathbf{V}^T \mathbf{a}_m \quad (58)$$

$$= \mathbf{a}_m^T \mathbf{V} \left(\mathbf{I} - \Sigma^T \left(\frac{1}{\text{SNR}} \mathbf{I} + \Sigma \Sigma^T \right)^{-1} \Sigma \right) \mathbf{V}^T \mathbf{a}_m \quad (59)$$

$$= \mathbf{a}_m^T \mathbf{V} \mathbf{D} \mathbf{V}^T \mathbf{a}_m. \quad (60)$$

Putting everything together, we have that

$$R(\mathbf{H}, \mathbf{a}_m) = -\frac{1}{2} \log \mathbf{a}_m^T \mathbf{V} \mathbf{D} \mathbf{V}^T \mathbf{a}_m. \quad (61)$$

■

C. Choosing Equations

In the previous section, we explored choices of the preprocessing matrix \mathbf{B} and characterized the optimal \mathbf{B} for a fixed full-rank integer matrix \mathbf{A} . Now, we discuss how to select equation coefficients $\mathbf{a}_1, \dots, \mathbf{a}_{2M}$ to maximize the achievable rate in Theorem 2 or, equivalently, Theorem 3. In the integer-forcing linear receiver, we are free to recover any full-rank set of linear equations with integer coefficients. However, due to the integer constraint on \mathbf{A} , it does not appear to be possible to give a closed-form solution for the best possible full-rank matrix \mathbf{A} .

An initially tempting choice for \mathbf{A} might be $\mathbf{A} = \mathbf{I}$. As we noted previously, for this choice of \mathbf{A} , selecting $\mathbf{B} = \mathbf{H}^\dagger$ reduces to the decorrelator while selecting $\mathbf{B} = \mathbf{H}^T \left(\frac{1}{\text{SNR}} \mathbf{I} + \mathbf{H} \mathbf{H}^T \right)^{-1}$ yields the linear MMSE estimator. However, as we show, for most channel matrices, fixing $\mathbf{A} = \mathbf{I}$ is suboptimal.

From Theorem 3, the achievable rate under the fixed channel matrix $\mathbf{A} = [\mathbf{a}_1, \dots, \mathbf{a}_{2M}]^T$ is given by

$$R < \max_{|\mathbf{A}| \neq 0} \min_m \left(-M \log \mathbf{a}_m^T \mathbf{V} \mathbf{D} \mathbf{V}^T \mathbf{a}_m \right). \quad (62)$$

In general, for a fixed SNR and channel matrix, finding the best coefficient matrix \mathbf{A} appears to be a combinatorial problem, requiring an explicit search over all possible full-rank integer matrices. The following lemma shows how the search space can be somewhat reduced.

Lemma 1: To optimize the achievable rate in Theorem 3 (or, equivalently, in Theorem 2), it is sufficient to check the space of all integer vectors \mathbf{a}_m with norm satisfying

$$\|\mathbf{a}_m\|^2 \leq 1 + \lambda_{\text{MAX}}^2 \text{SNR}. \quad (63)$$

where λ_{MAX} is the maximum singular value of \mathbf{H} .

Remark 5: This lemma thus shows that an exhaustive search only needs to check roughly SNR^M possibilities.

Proof: From (62), the achievable rate of the integer-forcing receiver is zero for all \mathbf{a}_m satisfying

$$\mathbf{a}_m^T \mathbf{V} \mathbf{D} \mathbf{V}^T \mathbf{a}_m \geq 1 \quad (64)$$

The left-hand side is lower bounded by

$$\mathbf{a}_m^T \mathbf{V} \mathbf{D} \mathbf{V}^T \mathbf{a}_m = \|\mathbf{D}^{1/2} \mathbf{V}^T \mathbf{a}_m\|^2 \quad (65)$$

$$= \sum_{i=1}^{2M} D_{i,i} |\mathbf{v}_i^T \mathbf{a}_m|^2 \quad (66)$$

$$\geq \min_i D_{i,i} \|\mathbf{a}_m\|^2 \quad (67)$$

$$= \frac{1}{1 + \lambda_{\text{MAX}}^2 \text{SNR}} \|\mathbf{a}_m\|^2 \quad (68)$$

Hence, if $\|\mathbf{a}_m\|^2 \geq 1 + \lambda_{\text{MAX}}^2 \text{SNR}$, then $\mathbf{a}_m^T \mathbf{V} \mathbf{D} \mathbf{V}^T \mathbf{a}_m \geq 1$. ■

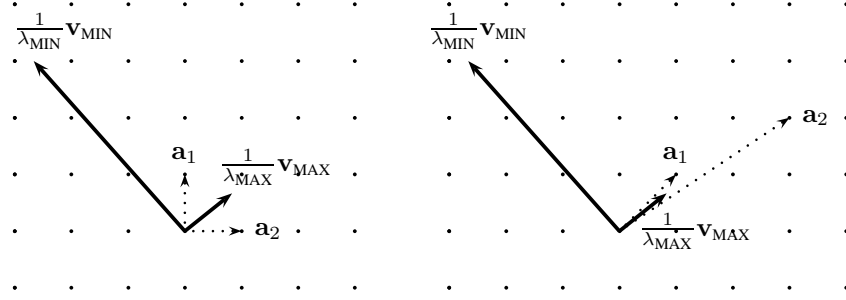


Fig. 4. The decorrelator (left) fixes the equations to be $\mathbf{a}_1 = [1 \ 0]^T$ and $\mathbf{a}_2 = [0 \ 1]^T$. The integer-forcing linear receiver (right) allows for any choice of linearly independent equations. Equations should be chosen in the direction of \mathbf{v}_{MAX} to avoid noise amplification by $\frac{1}{\lambda_{\text{MIN}}}$.

To conclude this subsection, we will now explicitly show how and why the choice $\mathbf{A} = \mathbf{I}$ is indeed suboptimal in general. In this context, it is instructive to use Lemma 1 to restate (62) as

$$R < \max_{\substack{|\mathbf{A}| \neq 0 \\ \|\mathbf{a}_m\|^2 \leq 1 + \lambda_{\text{MAX}}^2 \text{SNR}}} \min_m \left(-M \log \mathbf{a}_m^T \mathbf{V} \mathbf{D} \mathbf{V}^T \mathbf{a}_m \right). \quad (69)$$

Equation (69) suggests that we should choose coefficient vectors $\mathbf{a}_1, \dots, \mathbf{a}_{2M}$ to be short and in the direction of the maximum right eigenvector of \mathbf{H} . To make this concrete, let us study a particular 2×2 real MIMO channel for which the matrix \mathbf{H} has singular values λ_{MIN} and λ_{MAX} , with corresponding right singular vectors \mathbf{v}_{MIN} and \mathbf{v}_{MAX} , respectively, as illustrated in Figure 4. Here, decoder 1 recovers a linear combination of the transmitted message vectors with integer coefficients $\mathbf{a}_1 = [a_{1,1} \ a_{1,2}]^T$ and decoder 2 recovers a linear combination with integer coefficients $\mathbf{a}_2 = [a_{2,1} \ a_{2,2}]^T$. Using the exact integer-forcing rate from Corollary 1, the following rate is achievable

$$R < \min_{m=1,2} \log \left(\frac{\text{SNR}}{\tilde{\sigma}_m^2} \right) \quad (70)$$

where $\tilde{\sigma}_m^2$ can be interpreted as the effective noise variance for the m^{th} decoder,

$$\tilde{\sigma}_m^2 = \frac{1}{\lambda_{\text{MIN}}^2} |\mathbf{v}_{\text{MIN}}^T \mathbf{a}_m|^2 + \frac{1}{\lambda_{\text{MAX}}^2} |\mathbf{v}_{\text{MAX}}^T \mathbf{a}_m|^2. \quad (71)$$

Since $\frac{1}{\lambda_{\text{MIN}}} \geq \frac{1}{\lambda_{\text{MAX}}}$, (71) suggests that $\mathbf{a}_1, \mathbf{a}_2$ should be chosen in the direction of \mathbf{v}_{MAX} subject to linearly independent constraints to reduce the noise amplification by $\frac{1}{\lambda_{\text{MIN}}}$. In the case of the decorrelator (or MMSE receiver), the equation coefficients are fixed to be $\mathbf{a}_1 = [1 \ 0]^T$ and $\mathbf{a}_2 = [0 \ 1]^T$. As a result, the noise variance in at least one of the streams will be heavily amplified by $\frac{1}{\lambda_{\text{MIN}}}$ and the rate will be limited by the minimum singular value of the channel matrix. With integer-forcing, we are free to choose *any linearly independent* $\mathbf{a}_1, \mathbf{a}_2$ since we only require that our coefficients matrix \mathbf{A} be invertible. By choosing $\mathbf{a}_1, \mathbf{a}_2$ in the direction \mathbf{v}_{MAX} , we are protected against large noise amplification in the case of near-singular channel matrices.

D. Complexity

Our architecture has the same implementation complexity as that of a traditional linear receiver with the addition of the matrix search for \mathbf{A} . The ideal joint ML receiver aggregates the time and space dimensions and finds the ML estimate across both. As a result, its complexity is exponential in the product of the blocklength and the number of data streams. Our architecture decouples the time and space dimensions by allowing for single-stream decoding. First, we search for the best integer matrix \mathbf{A} , which has an exponential complexity in the number of data streams in the worst case. For slow fading channels, this search is only needed once per data frame. Afterwards, our receiver recovers M linearly independent equations of codewords according to \mathbf{A} and then solves these for the original codewords. This step is polynomial in the number of data streams and exponential in the blocklength for an ML decoder. In practice, the decoding step can be considerably accelerated through the use of LDPC codes and the integer matrix search can be sped up via a sphere decoder.

V. FIXED CHANNEL MATRICES

In this section, we compare the performance of the integer-forcing linear receiver against existing architectures through a series of examples. In Example 1, we compare the performance of different architectures for an ill-conditioned channel matrix and demonstrate that the choice of equation coefficients for the integer-forcing receiver changes with SNR. In Example 2, we compare the performance of the integer-forcing receiver with the decorrelator and show that the decorrelator can perform arbitrarily worse. In Examples 3, we illustrate that the gap between the integer-forcing receiver and lattice reduction can become unbounded. Finally, we show that the gap between the integer-forcing receiver and the joint decoder can be arbitrarily large in Example 4.

A. Example 1

Consider the 2×2 real MIMO channel with channel matrix

$$\mathbf{H} = \begin{bmatrix} 0.7 & 1.3 \\ 0.8 & 1.5 \end{bmatrix}. \quad (72)$$

Figure 5 shows the performance of the different architectures. (Recall that we assume equal-rate data streams on

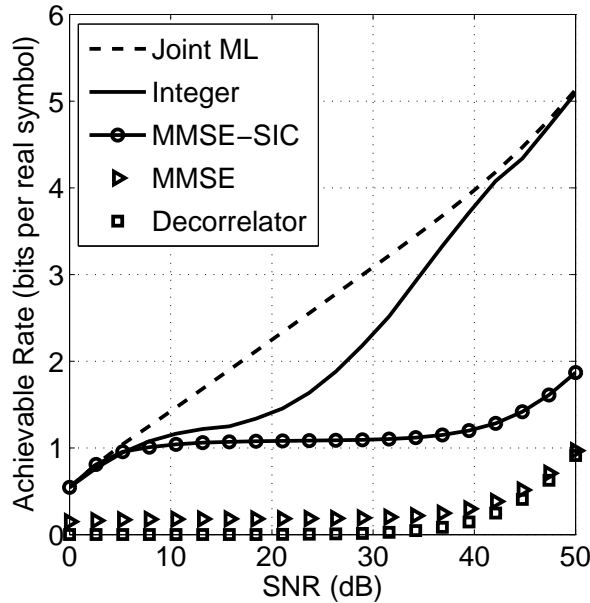


Fig. 5. Achievable rates for the 2×2 real-valued MIMO channel with fixed channel matrix from (72).

both transmit antennas, as in Definition 3.) The achievable rates for traditional linear receiver are given by (12) and that of the joint receiver is given by (8). The decorrelator and the MMSE receiver aim to separate the data streams and cancel the interference from other streams. However, this is difficult since the columns of the channel matrix are far from orthogonal. The integer-forcing architecture attempts to exploit the interference by decoding two linearly independent equations in the direction of the maximum eigenvector $\mathbf{v}_{\text{MAX}} = [0.47 \ 0.88]^T$. For example, at $\text{SNR} = 30\text{dB}$, we choose equation coefficients $\mathbf{a}_1 = [1 \ 2]^T$ and $\mathbf{a}_2 = [6 \ 11]^T$, while for $\text{SNR} = 40\text{dB}$, we choose equation coefficients $\mathbf{a}_1 = [1 \ 7]^T$ and $\mathbf{a}_2 = [2 \ 13]^T$. Thus, for different values of SNR, the optimal equation coefficients generally change.

B. Example 2: Integer-forcing vs. decorrelator

Consider the 2×2 real MIMO channel with channel matrix:

$$\mathbf{H} = \begin{bmatrix} 1 & 1 + \sqrt{\epsilon} \\ 0 & \epsilon \end{bmatrix} \quad (73)$$

where we assume $0 < \epsilon \ll 1$, $\frac{1}{\sqrt{\epsilon}}$ is an integer and $\text{SNR} \gg 1$. We first note that

$$\mathbf{H}^{-1} = \frac{1}{\epsilon} \begin{bmatrix} \epsilon & -(1 + \sqrt{\epsilon}) \\ 0 & 1 \end{bmatrix}, \quad (74)$$

Using (12) with $\mathbf{B} = \mathbf{H}^{-1}$, the achievable rate of the decorrelator is

$$R_{\text{DECORR}} = 2 \min \left\{ \frac{1}{2} \log \left(1 + \frac{\epsilon^2 \text{SNR}}{\epsilon^2 + \epsilon + 2\sqrt{\epsilon} + 1} \right), \frac{1}{2} \log (1 + \epsilon^2 \text{SNR}) \right\} \quad (75)$$

$$\leq \log (1 + \epsilon^2 \text{SNR}) \quad (76)$$

We compare the achievable rate of the decorrelator with the exact integer-forcing rate from Corollary 1. The equation coefficients selected by the decoders are

$$\mathbf{a}_1 = [1 \ 1]^T \quad (77)$$

$$\mathbf{a}_2 = \left[\frac{1}{\sqrt{\epsilon}} \quad \frac{1}{\sqrt{\epsilon}} + 1 \right]^T. \quad (78)$$

Using Corollary 1, the achievable rate of exact integer-forcing with equations coefficients $\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2]^T$ is

$$R_{\text{INTEGER}} = 2 \min_{m=1,2} \frac{1}{2} \log \left(\frac{\text{SNR}}{\|(\mathbf{H}^T)^{-1} \mathbf{a}_m\|^2} \right). \quad (79)$$

$$= 2 \min \left\{ \frac{1}{2} \log \left(\frac{\text{SNR}}{1 + \frac{1}{\epsilon}} \right), \frac{1}{2} \log \left(\frac{\text{SNR}}{\frac{1}{\epsilon}} \right) \right\} \quad (80)$$

$$= \log \left(\frac{\text{SNR}}{1 + \frac{1}{\epsilon}} \right) \quad (81)$$

$$\geq \log \left(\frac{\text{SNR}}{\frac{2}{\epsilon}} \right) \quad (82)$$

$$= \log \left(\frac{\epsilon \text{SNR}}{2} \right) \quad (83)$$

where the inequality follows since $0 < \epsilon \ll 1$.

We compare the two linear architectures to the joint ML decoder whose achievable rate is given by (8). For $0 < \epsilon \ll 1$ and $\text{SNR} \gg 1$, the rate of the joint decoder is

$$R_{\text{JOINT}} = \frac{1}{2} \log \det (\mathbf{I} + \mathbf{H}\mathbf{H}^T \text{SNR}) \quad (84)$$

$$= \frac{1}{2} \log \left((1 + \text{SNR})(1 + \epsilon^2 \text{SNR}) + (1 + \sqrt{\epsilon})^2 \text{SNR} \right) \quad (85)$$

Finally, let us compare the three rates in the setting where $\text{SNR} \rightarrow \infty$, and where the parameter ϵ in our channel model tends to zero according⁵ to $\epsilon \sim \frac{1}{\sqrt{\text{SNR}}}$. In that special case, we can observe that

$$R_{\text{DECORR}} \sim 1 \quad (86)$$

$$R_{\text{INTEGER}} \sim \frac{1}{2} \log(\text{SNR}) \quad (87)$$

$$R_{\text{JOINT}} \sim \frac{1}{2} \log(\text{SNR}) \quad (88)$$

Hence, the loss from using the decorrelator instead of the integer-forcing receiver becomes unbounded in this regime as $\text{SNR} \rightarrow \infty$. Furthermore, the integer-forcing receiver achieves the same scaling as the joint decoder.

⁵Recall that $f(\text{SNR}) \sim g(\text{SNR})$ implies that $\lim_{\text{SNR} \rightarrow \infty} \frac{f(\text{SNR})}{g(\text{SNR})} = 1$.

C. Example 3: Integer-forcing vs. lattice-reduction

In this example, we illustrate the difference between the proposed integer-forcing architecture and the lattice-reduction receiver. First, we note that, unlike the integer-forcing receiver, the lattice-reduction receiver is not required to use a lattice code but it should use a constellation with regular spacing, such as PAM or QAM. However, the key difference is that the effective channel matrix for lattice-reduction receivers is restricted to be unimodular⁶ while the effective channel for integer-forcing receivers can be any full-rank integer matrix. In this example, we show that this restriction can result in an arbitrarily large performance penalty. We consider the $M \times M$ MIMO channel with channel matrix:

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ -1 & -1 & -1 & -1 & \cdots & -1 & 2 \end{bmatrix} \quad (89)$$

A simple calculation shows that the inverse of this channel matrix is given by

$$\mathbf{H}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \quad (90)$$

Since \mathbf{H}^{-1} has non-integer entries, \mathbf{H} is not unimodular. The coefficient matrix that maximizes the achievable rate for the exact integer-forcing receiver from Corollary 1 is

$$\mathbf{A}_{\text{INTEGER}} = \mathbf{H} ,$$

leading to an effective noise variance in each stream that satisfies

$$\sigma_{\text{INTEGER}}^2 = 1 . \quad (91)$$

By contrast, following the lattice-reduction receiver, we must ensure that the resulting effective channel matrix is unimodular. Using the fact that $(\mathbf{H}^T)^{-1}$ is a basis for the body-centered cubic (BCC) lattice, it can be shown that the best choice of matrix is

$$\mathbf{A}_{\text{UNIMODULAR}} = \mathbf{I} . \quad (92)$$

It follows that the effective noise variance in the worst stream is given by

$$\sigma_{\text{UNIMODULAR}}^2 = \min_m \| (\mathbf{H}^T)^{-1} \mathbf{a}_{\text{UNIMODULAR},m} \|^2 \quad (93)$$

$$= \max \{ M/4, 1 \} . \quad (94)$$

Hence, as the number of antennas becomes large ($M \rightarrow \infty$), restricting the effective matrix to be unimodular results in an arbitrarily large loss.

D. Example 4: Integer-forcing vs. joint ML decoder

Finally, we illustrate the point that the integer-forcing receiver can sometimes be arbitrarily worse than optimal joint decoding. To see this, we consider a 2×2 MIMO channel with the channel matrix:

$$\mathbf{H} = \begin{bmatrix} 1 & 1 \\ 0 & \epsilon \end{bmatrix} \quad (95)$$

⁶Recall that a matrix is *unimodular* if has integer entries and its inverse has integer entries.

where $0 < \epsilon < 1$. The rate attainable via joint ML decoding is

$$R_{\text{JOINT}} = \log((1 + 2\text{SNR})(1 + \epsilon^2\text{SNR}) - \epsilon^2\text{SNR}^2) \quad (96)$$

$$\geq \log(1 + 2\text{SNR}) . \quad (97)$$

We note that the inverse of the channel matrix is given by

$$\mathbf{H}^{-1} = \begin{bmatrix} 1 & -\frac{1}{\epsilon} \\ 0 & \frac{1}{\epsilon} \end{bmatrix} . \quad (98)$$

We bound the achievable rate of exact integer-forcing from Corollary 1 as follows

$$R_{\text{INTEGER}} = 2 \max_{\mathbf{A}: |\mathbf{A}| \neq 0} \min_{m=1,2} \frac{1}{2} \log \left(\frac{\text{SNR}}{\|(\mathbf{H}^T)^{-1} \mathbf{a}_m\|^2} \right) \quad (99)$$

$$= 2 \max_{\mathbf{A}: |\mathbf{A}| \neq 0} \min_{m=1,2} \frac{1}{2} \log \left(\frac{\text{SNR}}{a_{m,1}^2 + (a_{m,2} - a_{m,1})^2 \frac{1}{\epsilon^2}} \right) \quad (100)$$

$$\leq \max_{a_{m,2} \neq a_{m,1}} \log \left(\frac{\text{SNR}}{a_{m,1}^2 + (a_{m,2} - a_{m,1})^2 \frac{1}{\epsilon^2}} \right) \quad (101)$$

$$\leq \log(\epsilon^2 \text{SNR}) . \quad (102)$$

Let $\epsilon \sim \frac{1}{\sqrt{\text{SNR}}}$ and consider the regime $\text{SNR} \rightarrow \infty$. The gap between the optimal joint receiver and the integer-forcing linear receiver can be arbitrarily large. However, as we will see in Section VI, the average behavior of the integer-forcing linear receiver is close to the joint decoder under a Rayleigh fading distribution for medium to high SNR.

VI. PERFORMANCE FOR SLOW FADING CHANNELS

A. Model and Definitions

We now demonstrate that integer-forcing receiver nearly matches the performance of the joint ML decoder under a slow fading channel model. Since the integer-forcing receiver can mimic the operation performed by a zero-forcing or MMSE receiver (as well as decode messages via equations), it is not surprising that it offers higher rates. However, these architectures are often coupled with some form of SIC. We will show that the integer-forcing receiver outperforms the following standard SIC architectures:

- V-BLAST I: The receiver decodes and cancels the data streams in a predetermined order, irrespective of the channel realization. Each data stream has the same rate. See (14) for the rate expression.
- V-BLAST II: The receiver selects the decoding order separately for each channel realization in such a way as to maximize the effective SNR for the data stream that sees the worst channel. Each data stream has the same rate. See (15) for the rate expression.
- V-BLAST III: The receiver decodes and cancels the data streams in a predetermined order. The rate of each data stream is selected to maximize the sum rate. The rate expression is given in Section VI-B.

In Sections VI-C and VI-D, we compare these schemes through simulations as well as their diversity-multiplexing tradeoffs. For completeness, we also compare integer-forcing to an SIC architecture that allows for both variable decoding order and unequal rate allocation in Appendix A.

We adopt the standard quasi-static Rayleigh fading model where each element of the complex channel matrix is independent and identically distributed according to a circularly symmetric complex normal distribution of unit variance. The transmitter is only aware of the channel statistics while the receiver knows the exact channel realization. As a result, we will have to cope with some outage probability p_{OUT} .

Definition 7: Assume there exists an architecture that encodes each data stream at the same rate and achieves sum rate $R(\mathbf{H})$. For a target rate R , then the outage probability is defined as

$$p_{\text{OUT}}(R) = \Pr(R(\mathbf{H}) < R). \quad (103)$$

For a fixed probability $p \in (0, 1]$, we define the *outage rate* to be

$$R_{\text{OUT}}(p) = \sup\{R : p_{\text{OUT}}(R) \leq p\}. \quad (104)$$

B. Rate Allocation

We have assumed that each data stream is encoded at the same rate. This is optimal for linear receivers under isotropic fading. However, when SIC is used, rate allocation can be beneficial in an outage scenario. To compare the performance of integer-forcing to SIC with rate allocation, we consider V-BLAST III in this section. V-BLAST III performs SIC with a fixed decoding order and allows for rate allocation among the different data streams. Without loss of generality for Rayleigh fading, if we fix a decoding order, we may take it to be $\pi = (1, 2, \dots, 2M)$. From (13), the achievable rate for stream m is

$$R_{\pi(m)}(\mathbf{H}) = \frac{1}{2} \log \left(1 + \frac{\text{SNR} \|\mathbf{b}_m^T \mathbf{h}_{\pi(m)}\|^2}{\|\mathbf{b}_m\|^2 + \text{SNR} \sum_{i>m} \|\mathbf{b}_m^T \mathbf{h}_{\pi(i)}\|^2} \right). \quad (105)$$

Since the streams are decoded in order, the later streams will achieve higher rates on average than the earlier streams. V-BLAST III allocates lower rates to earlier streams and higher rates to later streams. We now generalize Definition 7 to include rate allocation.

Definition 8: Assume an architecture that achieves rate $R_m(\mathbf{H})$ in stream m . For a target rate R , the the outage probability is given by:

$$p_{\text{OUT}}(R) = \min_{\substack{R_1, \dots, R_{2M} \\ \sum_{m=1}^{2M} R_m \leq R}} \Pr \left(\bigcup_{m=1}^{2M} \{R_m(\mathbf{H}) < R_m\} \right). \quad (106)$$

For a fixed probability $p \in (0, 1]$, we define the *outage rate* to be:

$$R_{\text{OUT}}(p) = \sup \{R : p_{\text{OUT}}(R) \leq p\}. \quad (107)$$

C. Outage Behavior

We now compare the outage rate and probabilities for the receiver architectures discussed above. It is easy to see that the zero-forcing receiver performs strictly worse than the MMSE receiver and V-BLAST I performs strictly worse than V-BLAST II. We have chosen to omit these two architectures from our plots to avoid overcrowding. Figure 6 shows the 1 percent outage rate and Figure 7 shows the 5 percent outage rate. In both cases, the integer-forcing receiver nearly matches the rate of the joint ML receiver while the MMSE receiver achieves significantly lower performance. The SIC architectures with either an optimal decoding order, V-BLAST II, or an optimized rate allocation, V-BLAST III, improve the performance of the MMSE receiver significantly but still achieve lower rates than the integer-forcing receiver from medium SNR onwards. Our simulations suggest that the outage rate of the integer-forcing receiver remains within a small gap from the outage rate of the joint ML receiver. However, we recall from the example given in Subsection V-D that it is not true that the integer-forcing receiver is uniformly near-optimal for all fading realizations. Figure 8 shows the outage probability for the target sum rate $R = 6$. We note that the integer-forcing receiver achieves the same slope as the joint decoder. In the next subsection, we characterize the diversity-multiplexing tradeoff of the integer-forcing receiver and compare it with the diversity-multiplexing tradeoff of traditional architectures that are considered in [25]. We show that the integer-forcing receiver attains the optimal diversity multiplexing-tradeoff in the case where each transmit antenna sends an independent data stream.

D. Diversity-Multiplexing Tradeoff

The diversity-multiplexing tradeoff (DMT) provides a rough characterization of the performance of a MIMO transmission scheme at high SNR [25].

Definition 9: A family of codes is said to achieve spatial multiplexing gain r and diversity gain d if the total data rate and the average probability of error satisfy

$$\lim_{\text{SNR} \rightarrow \infty} \frac{R(\text{SNR})}{\log \text{SNR}} \geq r \quad (108)$$

$$\lim_{\text{SNR} \rightarrow \infty} \frac{\log P_e(\text{SNR})}{\log \text{SNR}} \leq -d. \quad (109)$$

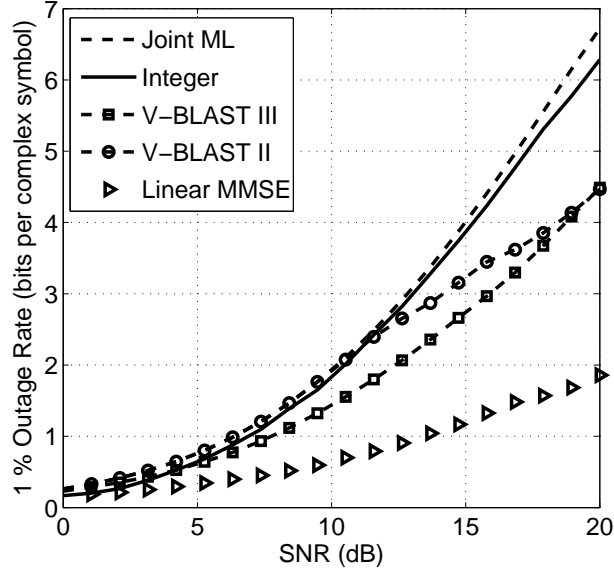


Fig. 6. 1 percent outage rates for the 2×2 complex-valued MIMO channel with Rayleigh fading.

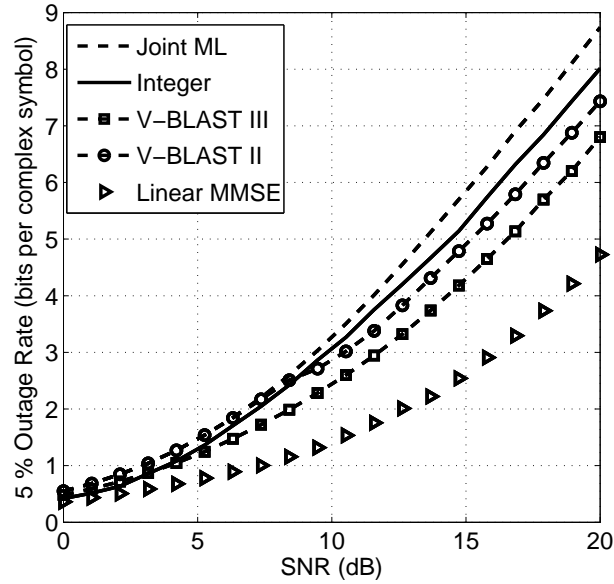


Fig. 7. 5 percent outage rates for the 2×2 complex-valued MIMO channel with Rayleigh fading.

In the case where each transmit antenna encodes an independent data stream⁷, the optimal DMT is

$$d_{\text{JOINT}}(r) = N \left(1 - \frac{r}{M} \right) \quad (110)$$

⁷If joint encoding across the antennas is permitted, then a better DMT is achievable. See [25] for more details.

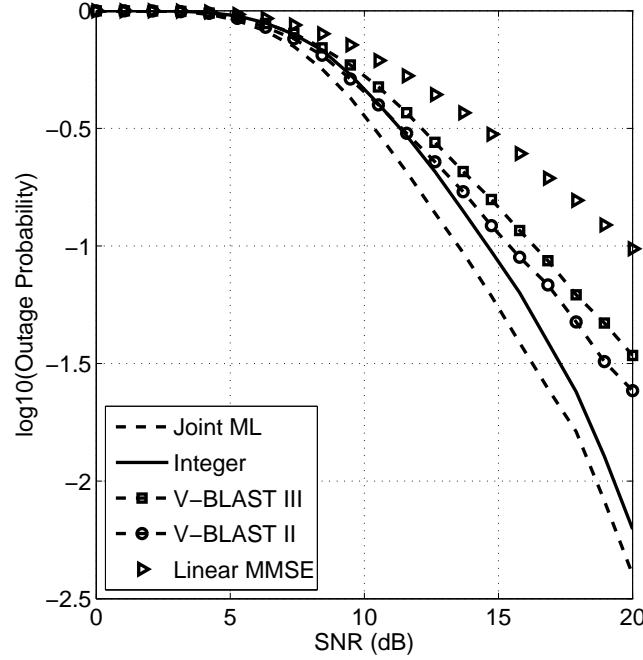


Fig. 8. Outage probability for the 2×2 complex-valued MIMO channel with Rayleigh fading for a target sum rate of $R = 6$.

where $r \in [0, M]$ and can be achieved by joint ML decoding [25]. The DMTs achieved by the decorrelator and SIC architectures are as follows [25]:

$$d_{\text{DECORR}}(r) = \left(1 - \frac{r}{M}\right) \quad (111)$$

$$d_{\text{V-BLAST I}}(r) = \left(1 - \frac{r}{M}\right) \quad (112)$$

$$d_{\text{V-BLAST II}}(r) \leq (N - 1) \left(1 - \frac{r}{M}\right) \quad (113)$$

$$d_{\text{V-BLAST III}}(r) = \text{piecewise linear curve connecting points } (r_k, n - k) \quad (114)$$

$$\text{where } r_0 = 0, r_k = \sum_{i=0}^{k-1} \frac{k-i}{n-i} \quad 1 \leq k \leq n$$

The decorrelator chooses the matrix \mathbf{B} to cancel the interference from the other data streams. As a result, the noise is heavily amplified when the channel matrix is near singular and the performance is limited by the minimum singular value of the channel matrix. In the integer-forcing linear receiver, the effective channel matrix \mathbf{A} is not limited to be the identity matrix but can be any full-rank integer matrix. This additional freedom is sufficient to recover the same DMT as the joint ML decoder.

Theorem 4: For a MIMO channel with M transmit, $N \geq M$ receive antennas, and Rayleigh fading, the achievable diversity-multiplexing tradeoff for the integer-forcing receiver is given by

$$d_{\text{INTEGER}}(r) = N \left(1 - \frac{r}{M}\right) \quad (115)$$

where $r \in [0, M]$.

The proof of Theorem 4 is given in Appendix B. Figure 9 illustrates the DMT for a 4×4 MIMO channel. The integer-forcing receiver achieves a maximum diversity of 4 while the decorrelator and V-BLAST I achieve can only achieve a diversity of 1 since their performance is limited by the worst data stream. V-BLAST II achieves a higher DMT than V-BLAST I but a lower diversity than the integer-forcing receiver since its rate is still limited by the worst stream after the optimal decoding order is applied. V-BLAST III achieves the optimal diversity at the

point $r = 0$ since only one data stream is used in transmission. For values of $r > 0$, the achievable diversity is suboptimal.

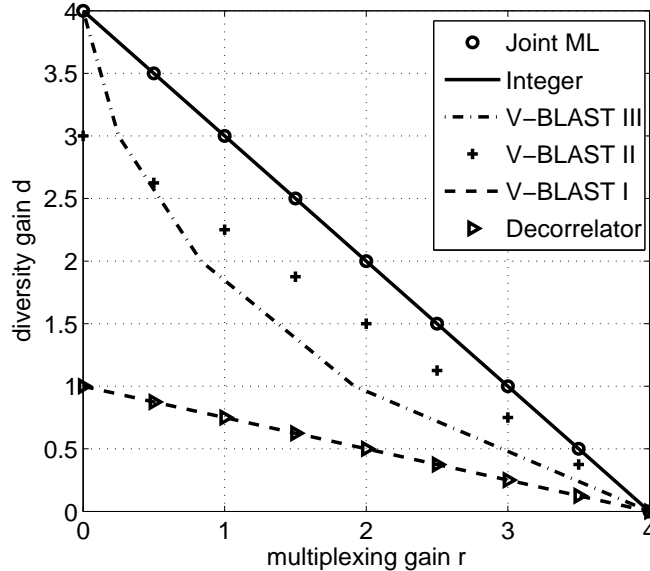


Fig. 9. Diversity-multiplexing tradeoff for the 4×4 MIMO channel with Rayleigh fading.

E. Discussion

As noted earlier, receiver architectures based on zero-forcing face a rate penalty when the channel matrix is ill-conditioned. Integer-forcing circumvents this issue by allowing the receiver to first decode equations whose coefficients are matched with those of the channel. From one perspective, the resulting gains are of a similar nature as those obtained by lattice-reduction receivers. One important difference is that integer-forcing applies a modulo operation at the receiver prior to decoding, which retains the linear structure of the codebook. This allows us to derive closed-form rate expressions analogous to those for traditional linear receiver. Typically, lattice reduction is used at the symbol level followed by a decoding step [26]. While this form of lattice reduction can be used to obtain the full receive diversity [27], it does not seem to suffice in terms of rate.

Another key advantage of integer-forcing is that it completely decouples the spatial aspect of decoding from the temporal aspect. That is, the search for the best integer matrix \mathbf{A} to approximate the channel matrix \mathbf{H} is completed before we attempt to decode the integer combinations of codewords. Thus, apart from the search⁸ for the best \mathbf{A} , which in a slow-fading environment does not have to be executed frequently, the complexity of the integer-forcing receiver is similar to that of the zero-forcing receiver.

From our outage plots, it is clear that the integer-forcing receiver significantly outperforms the basic MMSE receiver. Moreover, integer-forcing beats more sophisticated SIC-based V-BLAST architectures, even when these are permitted to optimize their rate allocation while integer-forcing is not. We note that it is possible to develop integer-forcing schemes that permit unequal rate allocations [6] as well as a form of interference cancellation [31] but this is beyond the scope of the present paper.

Integer-forcing also attains the full diversity-multiplexing tradeoff, unlike the V-BLAST architectures discussed above. Earlier work developed lattice-based schemes that attain the full DMT [28], [29] but, to the best of our knowledge, ours is the first that decouples spatial decoding from temporal decoding. The caveat is that the DMT result presented in the current paper only applies if there is no spatial coding across transmit antennas, whereas the DMT results of [28], [29] apply in general. Characterizing the DMT of integer-forcing when there is coding across transmit antennas is an interesting subject for future study.

⁸This search can be considerably sped up in practice through the use of a sphere decoding algorithm.

VII. OBLIVIOUS INTERFERENCE MITIGATION

A. MIMO Channel with Interference

We have studied the performance of the integer-forcing linear receiver under the standard MIMO channel and found that it achieves outage rates close those of the joint ML decoder as well as the same DMT. In this section, we show that integer-forcing architectures are also successful at dealing with a different kind of channel disturbance, namely interference. We assume that the interfering signal is low-dimensional (compared to the number of receive antennas), and we are most interested in the case where the variance of this interfering signal increases (at a certain rate) with the transmit power. We show that the integer-forcing architecture can be used to perform “oblivious” interference mitigation. By oblivious, we mean that the transmitter and receiver are unaware of the codebook of the interferer (if there is one). However, the receiver knows which subspace is occupied by the interference. By selecting equation coefficients in a direction that depends both on the interference space and on the channel matrix, the integer-forcing receiver reduces the impact of interference and attains a significant gain over traditional linear receivers. We will characterize the generalized degrees-of-freedom show that it matches that of the joint ML decoder.

Remark 6: Oblivious receivers have been thoroughly studied in the context of cellular systems [32] and distributed MIMO [33].

For ease of notation and tractability, the discussion presented in this section is limited to channels whose channel matrix is square, i.e., with equal number of transmit and receive antennas. Recall that the real-valued representation of the $M \times M$ complex-valued MIMO channel (see Definition 4) is given by

$$\mathbf{Y} = \mathbf{H}\mathbf{X} + \mathbf{Z} \quad (116)$$

where $\mathbf{Y} \in \mathbb{R}^{2M \times n}$ is the channel output, $\mathbf{H} \in \mathbb{R}^{2M \times 2M}$ is the real-valued representation of the fading matrix, $\mathbf{X}^{2M \times n}$ is the channel input, and the noise $\mathbf{Z} \in \mathbb{R}^{2M \times n}$ has i.i.d. Gaussian entries with unit variance. In this section, we extend the standard MIMO channel to include the case of interference. The generalized model has channel output

$$\mathbf{Y} = \mathbf{H}\mathbf{X} + \mathbf{J}\mathbf{V} + \mathbf{Z} \quad (117)$$

where \mathbf{H} is the channel matrix, \mathbf{X} is the channel input, and \mathbf{Z} is the noise, all as in the previous model. An external interferer adds $\mathbf{V} \in \mathbb{R}^{2K \times n}$ in the direction represented by the column space of $\mathbf{J} \in \mathbb{R}^{2M \times 2K}$. We assume that each element of \mathbf{V} is i.i.d. Gaussian with variance INR . We assume that \mathbf{J} is fixed during the whole transmission block and known only to the receiver.

The definition for messages, rates, encoders, and decoders follow along similar lines as those for the standard MIMO channel (see Definitions 1, 2, 5, and 3 in Section II).

B. Traditional Linear Receivers

As in the case without interference, traditional linear receivers process the channel output \mathbf{Y} by multiplying it by a $2M \times 2M$ matrix \mathbf{B} to arrive at the effective output

$$\tilde{\mathbf{Y}} = \mathbf{B}\mathbf{Y} \quad (118)$$

and recover the message \mathbf{w}_m using only the m^{th} row of the matrix $\tilde{\mathbf{Y}}$. By analogy to (12), the achievable sum rate can be expressed as

$$R_{\text{LINEAR}}(\mathbf{H}, \mathbf{J}, \mathbf{B}) = \min_m 2MR_m(\mathbf{H}, \mathbf{J}, \mathbf{B}). \quad (119)$$

where $R_m(\mathbf{H}, \mathbf{J}, \mathbf{B})$ represents the achievable rate for the m^{th} data stream (using Gaussian codebooks),

$$R_m(\mathbf{H}, \mathbf{J}, \mathbf{B}) = \frac{1}{2} \log \left(1 + \frac{\text{SNR} \|\mathbf{b}_m^T \mathbf{h}_m\|^2}{\|\mathbf{b}_m\|^2 + \text{INR} \|\mathbf{J}^T \mathbf{b}_m\|^2 + \text{SNR} \sum_{i \neq m} \|\mathbf{b}_m^T \mathbf{h}_i\|^2} \right),$$

Again, let us discuss several choices of the matrix \mathbf{B} . The decorrelator, given by $\mathbf{B} = \mathbf{H}^{-1}$, removes the interference due to other data streams but does not cancel the external interference \mathbf{J} (except in the very special case where the subspace spanned by \mathbf{J} is orthogonal to the subspace spanned by \mathbf{H}^{-1}). Alternatively, if we choose $\mathbf{B} = \mathbf{J}^\perp$, where \mathbf{J}^\perp is the $2K \times 2M$ matrix whose rowspace is orthogonal to the column space of \mathbf{J} , then the

external interference term is indeed nulled. The resulting output $\mathbf{J}^\perp \mathbf{Y}$ can then be processed by a traditional linear receiver. This scheme achieves good performance in high INR regimes but does not perform well in high SNR regimes since the interference due to the other data streams is mostly unresolved. The MMSE receiver improves the performance of the both architectures by choosing $\mathbf{B} = \mathbf{H} \left(\frac{1}{\text{SNR}} \mathbf{I} + \frac{\text{INR}}{\text{SNR}} \mathbf{J} \mathbf{J}^T + \mathbf{H} \mathbf{H}^T \right)^{-1}$. However, since there are $2M$ data streams and the interference is of dimension $2K$, it is impossible to cancel both the interference from other data streams and the external interference with any matrix \mathbf{B} . One way out of this conundrum is to reduce the number of transmitted streams to $2M - 2K$. For this scenario, the MMSE receiver can be applied to mitigate both the external interference and the interference from other data streams.

Complexity permitting, we can again improve performance by resorting to successive interference cancellation architectures. The achievable rate for V-BLAST I in the standard MIMO channel from (14) becomes

$$R_{\text{SIC},1}(\mathbf{H}) = \min_m 2MR_{\pi(m)}(\mathbf{H}). \quad (120)$$

where

$$R_{\pi(m)}(\mathbf{H}) = \frac{1}{2} \log \left(1 + \frac{\text{SNR} \|\mathbf{b}_m^T \mathbf{h}_{\pi(m)}\|^2}{\|\mathbf{b}_m\|^2 + \text{INR} \|\mathbf{J}^T \mathbf{b}_m\|^2 + \text{SNR} \sum_{i>m} \|\mathbf{b}_m^T \mathbf{h}_{\pi(i)}\|^2} \right) \quad (121)$$

and $\mathbf{b}_m = \left(\frac{1}{\text{SNR}} \mathbf{I} + \mathbf{J} \mathbf{J}^T \frac{\text{INR}}{\text{SNR}} + \mathbf{H}_{\pi_m} \mathbf{H}_{\pi_m}^T \right)^{-1} \mathbf{h}_{\pi(m)}$. The achievable rate for V-BLAST II follows by maximizing the rate in (120) over all decoding orders $\pi \in \Pi$.

C. Integer-Forcing Linear Receiver

We apply the integer-forcing linear receiver proposed in Section IV to the problem of mitigating interference (see Figure 10). The channel output matrix \mathbf{Y} is first multiplied by a fixed matrix \mathbf{B} to form the matrix $\tilde{\mathbf{Y}}$ whose m^{th} row is the signal fed into the m^{th} decoder. Each such row can be expressed as

$$\tilde{\mathbf{y}}_m^T = \sum_{i=1}^{2M} (\mathbf{b}_m^T \mathbf{h}_i) \mathbf{x}_i^T + \mathbf{b}_m^T \mathbf{J} \mathbf{V} + \mathbf{b}_m^T \mathbf{Z} \quad (122)$$

$$= \sum_{i=1}^{2M} \tilde{\mathbf{h}}_m^T \mathbf{X} + \tilde{\mathbf{v}}_m^T + \tilde{\mathbf{z}}_m^T \quad (123)$$

where $\tilde{\mathbf{h}}_m = \mathbf{H}^T \mathbf{b}_m$ is the effective channel to the m^{th} decoder, $\tilde{\mathbf{v}}_m$ is the effective interference with variance $\|\mathbf{J}^T \mathbf{b}_m\|^2 \text{INR}$, and $\tilde{\mathbf{z}}_m$ is the effective noise with variance $\|\mathbf{b}_m\|^2$. The next theorem and its following remarks generalize Theorem 2, Corollary 1, and Corollary 2 to include the case with interference.

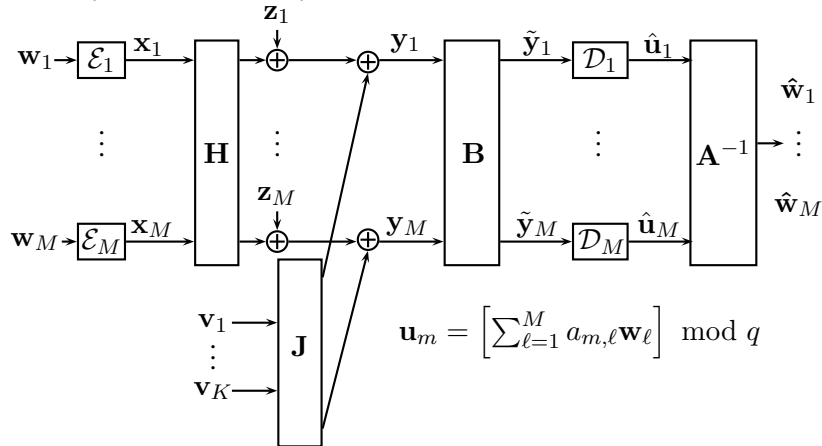


Fig. 10. Integer-forcing linear receiver for the complex-valued $M \times M$ MIMO channel with external inference of dimension K .

Theorem 5: Consider the MIMO channel with channel matrix $\mathbf{H} \in \mathbb{R}^{2M \times 2M}$ and interference matrix $\mathbf{J} \in \mathbb{R}^{2M \times 2K}$. For any full-rank integer matrix $\mathbf{A} \in \mathbb{Z}^{2M \times 2M}$ and any $2M \times 2M$ matrix $\mathbf{B} = [\mathbf{b}_1 \cdots \mathbf{b}_{2M}]^T$, the following sum rate is achievable using the integer-forcing linear receiver:

$$R(\mathbf{H}, \mathbf{J}, \mathbf{A}, \mathbf{B}) < \min_m 2MR(\mathbf{H}, \mathbf{J}, \mathbf{a}_m, \mathbf{b}_m) \quad (124)$$

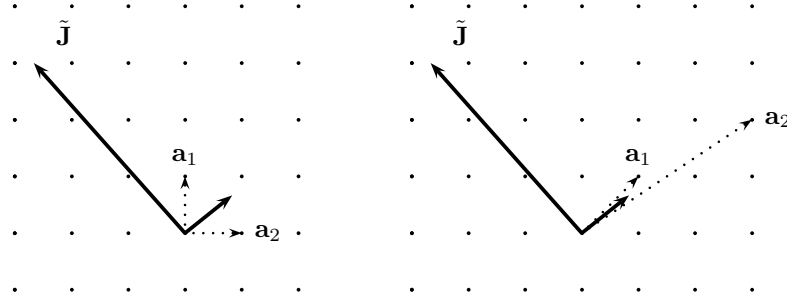


Fig. 11. The decorrelator (left) fixes the equations to be $\mathbf{a}_1 = [1 \ 0]^T$ and $\mathbf{a}_2 = [0 \ 1]^T$. The integer-forcing Linear Receiver (right) allows for any choice of linearly independent equations. Equations should be chosen in the direction orthogonal to $\tilde{\mathbf{J}} = \mathbf{H}^{-1}\mathbf{J}$.

where $R(\mathbf{H}, \mathbf{J}, \mathbf{a}_m, \mathbf{b}_m)$ is given by

$$= \frac{1}{2} \log \left(\frac{\text{SNR}}{\|\mathbf{b}_m\|^2 + \|\mathbf{J}^T \mathbf{b}_m\|^2 \text{INR} + \|\mathbf{H}^T \mathbf{b}_m - \mathbf{a}_m\|^2 \text{SNR}} \right)$$

Remark 7: Exact integer-forcing selects $\mathbf{B} = \mathbf{A}\mathbf{H}^{-1}$. The achievable rate can be expressed more concisely as

$$R < \min_m M \log \left(\frac{\text{SNR}}{\|(\mathbf{H}^{-1})^T \mathbf{a}_m\|^2 + \|\mathbf{J}^T (\mathbf{H}^{-1})^T \mathbf{a}_m\|^2 \text{INR}} \right). \quad (125)$$

Remark 8: The optimal projection matrix that maximizes the achievable rate in Theorem 5 is given by

$$\mathbf{B}_{\text{OPT}} = \mathbf{A}\mathbf{H}^T \left(\mathbf{H}\mathbf{H}^T + \mathbf{J}\mathbf{J}^T \frac{\text{INR}}{\text{SNR}} + \mathbf{I} \frac{1}{\text{SNR}} \right)^{-1}. \quad (126)$$

D. Geometric Interpretation

In the case without interference, the equation coefficients $\mathbf{a}_1, \dots, \mathbf{a}_M$ should be chosen in the direction of the maximum eigenvector of $\mathbf{H}^T \mathbf{H}$ to minimize the effective noise (see Figure 4). When the interference is large, the equations coefficients should instead be chosen as close to orthogonal to the effective interference as possible. Consider the (suboptimal) rate expression in (125). The “effective” noise variance in the m^{th} stream is

$$\sigma_{\text{EFFEC},m} = \|(\mathbf{H}^{-1})^T \mathbf{a}_m\|^2 + \|\mathbf{J}^T (\mathbf{H}^{-1})^T \mathbf{a}_m\|^2 \text{INR}. \quad (127)$$

Let λ_{MAX} be the maximum singular value of \mathbf{H}^{-1} and $\tilde{\mathbf{J}} = \mathbf{H}^{-1}\mathbf{J}$. The effective noise variance can be bounded by

$$\sigma_{\text{EFFEC},m} \leq \lambda_{\text{MAX}}^2 \|\mathbf{a}_m\|^2 + \|\tilde{\mathbf{J}}^T \mathbf{a}_m\|^2 \text{INR}. \quad (128)$$

In the high interference regime ($\text{INR} \gg 1$), the equation coefficients should be chosen orthogonal to the direction of the “effective” interference $\tilde{\mathbf{J}}$ to minimize the effective noise variance. This is illustrated in Figure 11. In the case of traditional linear receivers, the equation coefficients are fixed to be the unit vectors: $\mathbf{a}_1 = [1 \ 0 \ \dots \ 0]^T$, $\mathbf{a}_2 = [0 \ 1 \ \dots \ 0]^T, \dots, \mathbf{a}_{2M} = [0 \ 0 \ \dots \ 1]^T$. As a result, the interference space spanned by $\tilde{\mathbf{J}}$ has significant projections onto at least some of the decoding dimensions \mathbf{a}_m . By contrast, in the case of the integer-forcing linear receiver, since $\mathbf{a}_1, \dots, \mathbf{a}_{2M}$ need only be linearly independent, we can choose all of the decoding dimensions \mathbf{a}_m to be close to orthogonal to $\tilde{\mathbf{J}}$.

E. Fixed Channel Example

To illustrate the impact of choosing equation coefficients in a fashion suitable to mitigate external interference, we consider the 2×2 MIMO channel with channel matrix \mathbf{H} and one-dimensional interference space \mathbf{J} given by

$$\mathbf{H} = \frac{1}{3} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad \mathbf{J} = \frac{1}{3} \begin{bmatrix} L+2 \\ 2L+1 \end{bmatrix} \quad (129)$$

where $L \in \mathbb{N}$. In the case of the decorrelator, we invert the channel to arrive at the effective output:

$$\tilde{\mathbf{Y}} = \mathbf{X} + \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} L+2 \\ 2L+1 \end{bmatrix} \mathbf{V} + \tilde{\mathbf{Z}} \quad (130)$$

$$= \mathbf{X} + \begin{bmatrix} 1 \\ L \end{bmatrix} \mathbf{V} + \tilde{\mathbf{Z}} \quad (131)$$

where $\tilde{\mathbf{Z}} = \mathbf{H}^{-1}\mathbf{Z}$. For $\text{INR} \gg 1$, the effective noise variances scale as

$$\sigma_{\text{DECORR},1}^2 \sim \text{INR} \quad (132)$$

$$\sigma_{\text{DECORR},2}^2 \sim L^2 \text{INR} \quad (133)$$

Using the integer-forcing linear receiver with the choice of equations $\mathbf{a}_1 = [1 \ 0]^T$ and $\mathbf{a}_2 = [-L \ 1]^T$, the effective channel output to the second decoder is

$$[-L \ 1] \tilde{\mathbf{Y}} = -L\mathbf{x}_1^T + \mathbf{x}_2^T + [-L \ 1] \tilde{\mathbf{Z}} \quad (134)$$

where \mathbf{x}_ℓ is the codeword sent by the ℓ^{th} antenna. It follows that the effective noise variances are

$$\sigma_{\text{INT},1}^2 \sim \text{INR} \quad (135)$$

$$\sigma_{\text{INT},2}^2 \sim C \quad (136)$$

where C is a constant that does not scale with INR . In this example, the integer-forcing linear receiver is able to completely cancel the effect of interference in the second stream by choosing equation coefficients appropriately.

F. Generalized Degrees of Freedom

We evaluate the generalized degrees of freedom for the $M \times M$ complex MIMO channel with K -dimensional interference. We specify the interference-to-noise ratio through the parameter α where

$$\alpha = \lim_{\substack{\text{SNR} \rightarrow \infty \\ \text{INR} \rightarrow \infty}} \frac{\log \text{INR}}{\log \text{SNR}} \quad (137)$$

and consider the case where $0 \leq \alpha \leq 1$. The generalized degrees of freedom are defined as follows (see [34]):

Definition 10: (Generalized Degrees-of-Freedom) For a given channel matrix \mathbf{H} and interference matrix \mathbf{J} , the generalized degrees-of-freedom of a scheme is

$$d(\mathbf{H}, \mathbf{J}) = \lim_{\substack{\text{SNR} \rightarrow \infty \\ \text{INR} = \text{SNR}^\alpha}} \frac{R(\text{SNR}, \mathbf{H}, \mathbf{J})}{\log \text{SNR}} \quad (138)$$

where $R(\text{SNR}, \mathbf{H}, \mathbf{J})$ is the achievable sum rate of the scheme.

Definition 11: (Rational Independence) We call a matrix $\mathbf{T}^{M \times N}$ rationally independent if for all $\mathbf{q} \in \mathbb{Q}^N$, we have that

$$\mathbf{T}\mathbf{q} \neq \mathbf{0} \quad (139)$$

We consider the set of matrices (\mathbf{H}, \mathbf{J}) such that $\mathbf{H}^{-1}\mathbf{J}$ is rationally independent. It can be seen that this set has Lebesgue measure one. In the next theorem, we show that for this class of matrices, the integer-forcing linear receiver achieves the same number of generalized degrees of freedom as the joint decoder, and is thus optimal.

Theorem 6: Consider the $M \times M$ complex MIMO channel with K dimensional interference. The integer-forcing linear receiver achieves the generalized degrees of freedom

$$d_{\text{INT}} = M - K\alpha \quad (140)$$

for a set of \mathbf{H}, \mathbf{J} that have Lebesgue measure one.

A straightforward derivation shows that the optimal joint decoder with $2M$ streams of data achieves the following generalized degrees of freedom

$$d_{\text{JOINT}} = M - K\alpha \quad (141)$$

$$(142)$$

for all full-rank channel matrices \mathbf{H}, \mathbf{J} . The linear MMSE receiver with $2M$ data streams and the linear MMSE receiver with $2M - 2K$ data streams achieve the following degrees of freedom for all full-rank channel matrices \mathbf{H}, \mathbf{J} :

$$d_{\text{MMSE}, 2M} = M - M\alpha \quad (143)$$

$$d_{\text{MMSE}, 2M-2K} = M - K \quad (144)$$

When $2M$ data streams are transmitted (on the real-valued representation of the complex MIMO channel), the MMSE receiver does not achieve the optimal number of degrees of freedom since it treats the interference as noise at high SNR while the integer-forcing linear receiver mitigates the interference. When only $2M - 2K$ data streams are transmitted, the linear MMSE receiver can first cancel the interference and then separate the data streams to achieve a degree of freedom of $2M - 2K$. However, this is suboptimal for all regimes $\alpha < 1$ (see Figure 12). A straightforward calculation shows that when the number of transmitted data streams is between $2M - 2K$ and $2M$, the performance is strictly suboptimal in terms of degrees of freedom.

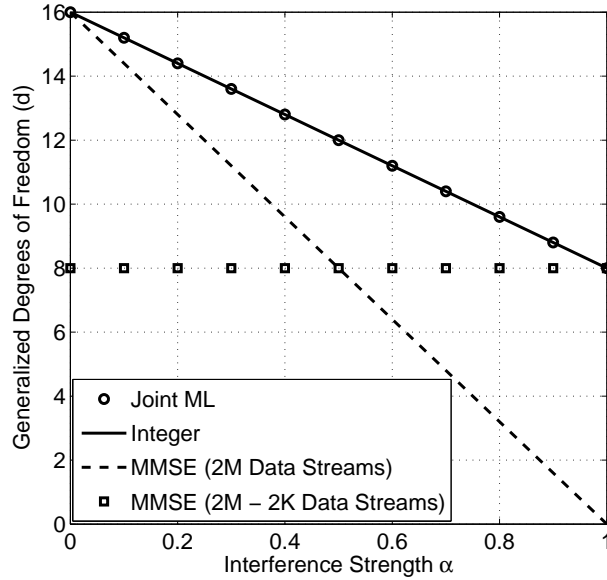


Fig. 12. Generalized degrees of freedom for the complex-valued 16×16 MIMO channel with 8-dimensional interference ($M = 16, K = 8$).

Our proof of Theorem 6 uses the following lemma. Recall that a set of vectors is linearly independent if none of the vectors can be written as a linear combination of the others.

Lemma 2: Let $\mathbf{T} \in \mathbb{R}^{2K \times (2M-2K)}$ be rationally independent and assume $|t_{i,j}| \leq 1$ for all i, j . There exists a $Q' \in \mathbb{N}$ such that for all $Q > Q'$, there exist $2M$ linearly independent integer vectors $[\mathbf{q}_m^T, \mathbf{p}_m^T]^T \in \mathbb{Z}^{2M-2K} \times \mathbb{Z}^{2K}$ for $m = 1, 2, \dots, 2M$, satisfying

$$\|\mathbf{q}_m\| \leq CQ(\log Q)^2 \quad (145)$$

$$\|\mathbf{T}\mathbf{q}_m - \mathbf{p}_m\| \leq \frac{C(\log Q)^2}{Q^{\frac{M-K}{K}}}, \quad (146)$$

where C is a constant that is independent of Q .

Lemma 2 bounds the approximation of \mathbf{T} with $2M$ linearly independent integer vectors. Its proof is given in Appendix C. We now prove Theorem 6.

Proof: (Theorem 6) To establish this result, we use the (generally suboptimal) choice of the matrix \mathbf{B} that we have referred to as exact integer-forcing, i.e., from (125). The achievable rate of this version of the integer-forcing linear receiver is given by

$$R < \max_{\mathbf{A}: |\mathbf{A}| \neq 0} \min_m M \log \left(\frac{\text{SNR}}{\|(\mathbf{H}^{-1})^T \mathbf{a}_m\|^2 + \|\mathbf{J}^T (\mathbf{H}^{-1})^T \mathbf{a}_m\|^2 \text{INR}} \right). \quad (147)$$

With $\text{INR} \sim \text{SNR}^\alpha$, the effective noise in the worst data stream can be expressed as

$$\sigma^2 = \min_{\mathbf{A}: |\mathbf{A}| \neq 0} \max_m \|(\mathbf{H}^{-1})^T \mathbf{a}_m\|^2 + \|\mathbf{J}^T (\mathbf{H}^{-1})^T \mathbf{a}_m\|^2 \text{SNR}^\alpha \quad (148)$$

$$\leq \min_{\mathbf{A}: |\mathbf{A}| \neq 0} \max_m \lambda_{\text{MAX}}^2(\mathbf{H}^{-1}) \|\mathbf{a}_m\|^2 + \|\tilde{\mathbf{J}}^T \mathbf{a}_m\|^2 \text{SNR}^\alpha, \quad (149)$$

where $\lambda_{\text{MAX}}(\mathbf{H}^{-1})$ is the maximum singular value of \mathbf{H}^{-1} and we use the shorthand $\tilde{\mathbf{J}} = \mathbf{H}^{-1} \mathbf{J}$. Let us partition $\tilde{\mathbf{J}}^T$ into two parts,

$$\tilde{\mathbf{J}}^T = [\mathbf{S}_1, \mathbf{S}_2], \quad (150)$$

where $\mathbf{S}_1 \in \mathbb{R}^{2K \times (2M-2K)}$ and $\mathbf{S}_2 \in \mathbb{R}^{2K \times 2K}$. Observe that with probability one, $\tilde{\mathbf{J}}^T$ has rank $2K$ (hence, is full-rank). This implies that we can permute the columns of $\tilde{\mathbf{J}}^T$ in such a way as to ensure that its last $2K$ columns are linearly independent. If we use the same permutation on the coefficients of the vector \mathbf{a}_m , our upper bound on the effective noise variance given in (149) will remain unchanged. Therefore, without loss of generality, we may assume that \mathbf{S}_2 has rank $2K$. We define $\mathbf{T} = -\mathbf{S}_2^{-1} \mathbf{S}_1$. Then, we can write

$$\mathbf{S}_2^{-1} \tilde{\mathbf{J}}^T = [\mathbf{S}_2^{-1} \mathbf{S}_1, \mathbf{S}_2^{-1} \mathbf{S}_2] \quad (151)$$

$$= [-\mathbf{T}, \mathbf{I}_{2K}]. \quad (152)$$

Let the coefficients for the m^{th} equation be given by $\mathbf{a}_m = [\mathbf{q}_m^T, \mathbf{p}_m^T]^T$ where $\mathbf{q}_m \in \mathbb{Z}^{2M-2K}$ and $\mathbf{p}_m \in \mathbb{Z}^{2K}$. We use (152) to bound $\|\tilde{\mathbf{J}}^T \mathbf{a}_m\|$:

$$\|\tilde{\mathbf{J}}^T \mathbf{a}_m\|^2 = \|\mathbf{S}_2 \mathbf{S}_2^{-1} \tilde{\mathbf{J}}^T \mathbf{a}_m\|^2 \quad (153)$$

$$= \|\mathbf{S}_2 [-\mathbf{T}, \mathbf{I}_{2K}] \mathbf{a}_m\|^2 \quad (154)$$

$$\leq \lambda_{\text{MAX}}^2(\mathbf{S}_2) \|[-\mathbf{T}, \mathbf{I}_{2K}] \mathbf{a}_m\|^2 \quad (155)$$

$$= \lambda_{\text{MAX}}^2(\mathbf{S}_2) \|\mathbf{T} \mathbf{q}_m - \mathbf{p}_m\|^2 \quad (156)$$

where $\lambda_{\text{MAX}}(\mathbf{S}_2)$ the maximum singular value of \mathbf{S}_2 . Combining (156) with (149), the effective noise variance is bounded as follows:

$$\sigma^2 \leq \min_{\mathbf{A}: |\mathbf{A}| \neq 0} \max_m \lambda_{\text{MAX}}^2(\mathbf{H}^{-1}) \|\mathbf{a}_m\|^2 + \lambda_{\text{MAX}}^2(\mathbf{S}_2) \|\mathbf{T} \mathbf{q}_m - \mathbf{p}_m\|^2 \text{SNR}^\alpha. \quad (157)$$

We proceed to bound the quantity $\|\mathbf{T} \mathbf{q}_m - \mathbf{p}_m\|^2$. We decompose \mathbf{T} into its integer and fractional parts:

$$\mathbf{T} = \mathbf{T}_I + \mathbf{T}_F \quad (158)$$

where \mathbf{T}_I represents the integer part of \mathbf{T} and \mathbf{T}_F represents the fractional part of \mathbf{T} . We define

$$\tilde{\mathbf{p}}_m = \mathbf{p}_m - \mathbf{T}_I \mathbf{q}_m \quad (159)$$

$$\tilde{\mathbf{a}}_m = [\mathbf{q}_m^T, \tilde{\mathbf{p}}_m^T]^T \quad (160)$$

$$\tilde{\mathbf{A}} = [\tilde{\mathbf{a}}_1 \cdots \tilde{\mathbf{a}}_{2M}]^T \quad (161)$$

$$\mathbf{G} = \begin{pmatrix} \mathbf{I}_{2M-2K} & \mathbf{0} \\ -\mathbf{T}_I & \mathbf{I}_{2K} \end{pmatrix}. \quad (162)$$

Since \mathbf{G} is a $2M \times 2M$ lower triangular matrix with non-zero diagonal elements, it has rank $2M$. We note that

$$\mathbf{a}_m = \mathbf{G}^{-1} \tilde{\mathbf{a}}_m. \quad (163)$$

Since \mathbf{G} is invertible, it follows that if the matrix formed by the coefficient vectors $\mathbf{a}_1, \dots, \mathbf{a}_{2M}$ is full-rank, then the matrix formed by $\tilde{\mathbf{a}}_1, \dots, \tilde{\mathbf{a}}_{2M}$ is full-rank and vice versa. From (158), it have that

$$\|\mathbf{T} \mathbf{q}_m - \mathbf{p}_m\| = \|\mathbf{T}_F \mathbf{q}_m - (\mathbf{p}_m - \mathbf{T}_I \mathbf{q}_m)\| \quad (164)$$

$$= \|\mathbf{T}_F \mathbf{q}_m - \tilde{\mathbf{p}}_m\| \quad (165)$$

and, from (163), we have that

$$\|\mathbf{a}_m\|^2 = \|\mathbf{G}^{-1}\tilde{\mathbf{a}}_m\|^2 \quad (166)$$

$$\leq \lambda_{\text{MAX}}^2(\mathbf{G}^{-1})\|\tilde{\mathbf{a}}_m\|^2. \quad (167)$$

From (157), (165), and (167), the effective noise variance can be upper bounded by

$$\sigma^2 \leq \min_{\tilde{\mathbf{A}}:|\tilde{\mathbf{A}}|\neq 0} \max_m \lambda_{\text{MAX}}^2(\mathbf{H}^{-1})\lambda_{\text{MAX}}^2(\mathbf{G}^{-1})\|\tilde{\mathbf{a}}_m\|^2 + \lambda_{\text{MAX}}^2(\mathbf{S}_2)\|\mathbf{T}_F\mathbf{q}_m - \tilde{\mathbf{p}}_m\|^2 \text{SNR}^\alpha \quad (168)$$

From Lemma 2, there exists a Q' such that for all $Q > Q'$, there exist $2M$ linearly independent vectors:

$$\tilde{\mathbf{a}}_m = [\mathbf{q}_m^T, \tilde{\mathbf{p}}_m^T]^T \in \mathbb{Z}^{2M-2K} \times \mathbb{Z}^{2K} \quad \text{for } m = 1, \dots, 2M \quad (169)$$

satisfying the following two inequalities:

$$\|\mathbf{q}_m\| \leq C(\log Q)^2 Q \quad (170)$$

$$\|\mathbf{T}_F\mathbf{q}_m - \tilde{\mathbf{p}}_m\| \leq \frac{C(\log Q)^2}{Q^{\frac{M-K}{K}}} \quad (171)$$

where C is some constant independent of Q . For sufficiently large Q , we observe that $\frac{C(\log Q)^2}{Q^{\frac{M-K}{K}}} \leq 1$. Using (170) and (171), we bound the norm of $\tilde{\mathbf{a}}_m$ as follows:

$$\|\tilde{\mathbf{a}}_m\| = \sqrt{\|\mathbf{q}_m\|^2 + \|\tilde{\mathbf{p}}_m\|^2} \quad (172)$$

$$\leq \|\mathbf{q}_m\| + \|\tilde{\mathbf{p}}_m\| \quad (173)$$

$$= \|\mathbf{q}_m\| + \|\tilde{\mathbf{p}}_m + \mathbf{T}_F\mathbf{q}_m - \mathbf{T}_F\mathbf{q}_m\| \quad (174)$$

$$\leq \|\mathbf{q}_m\| + \|\mathbf{T}_F\mathbf{q}_m\| + \|\tilde{\mathbf{p}}_m - \mathbf{T}_F\mathbf{q}_m\| \quad (175)$$

$$\leq \|\mathbf{q}_m\| + \|\mathbf{T}_F\mathbf{q}_m\| + \frac{C(\log Q)^2}{Q^{\frac{M-K}{K}}} \quad (176)$$

$$\leq \|\mathbf{q}_m\| + \|\mathbf{T}_F\mathbf{q}_m\| + 1 \quad (177)$$

$$\leq \|\mathbf{q}_m\| + \lambda_{\text{MAX}}(\mathbf{T}_F)\|\mathbf{q}_m\| + 1 \quad (178)$$

$$\leq C(\log Q)^2 Q(1 + \lambda_{\text{MAX}}(\mathbf{T}_F)) + 1 \quad (179)$$

where $\lambda_{\text{MAX}}(\mathbf{T}_F)$ is the the maximum singular value of \mathbf{T}_F .

Combining (171) and (179), the effective noise variance from (168) is bounded by

$$\sigma^2 \leq \lambda_{\text{MAX}}^2(\mathbf{H}^{-1})\lambda_{\text{MAX}}^2(\mathbf{G}^{-1})\left(C(\log Q)^2 Q(1 + \lambda_{\text{MAX}}(\mathbf{T})) + 1\right)^2 + \lambda_{\text{MAX}}^2(\mathbf{S}_2)\text{SNR}^\alpha \left(\frac{C(\log Q)^2}{Q^{\frac{M-K}{K}}}\right)^2 \quad (180)$$

Let Q scale according to $Q^2 \sim \text{SNR}^\gamma$. It follows that

$$\sigma^2 \leq \Theta(\log \text{SNR}) \left(\text{SNR}^\gamma + \text{SNR}^{\alpha - \gamma(\frac{M-K}{K})} \right). \quad (181)$$

Setting $\gamma = \frac{K}{M}\alpha$, we find that the generalized degrees-of-freedom are

$$d_{\text{INT}} = \lim_{\text{SNR} \rightarrow \infty} 2M \frac{\frac{1}{2} \log \left(\frac{\text{SNR}}{\sigma^2} \right)}{\log \text{SNR}} \quad (182)$$

$$= \lim_{\text{SNR} \rightarrow \infty} M \frac{\log \left(\frac{\text{SNR}}{\text{SNR}^{\alpha \frac{K}{M}}} \right)}{\log \text{SNR}} \quad (183)$$

$$= M \left(1 - \alpha \frac{K}{M} \right) \quad (184)$$

$$= M - K\alpha, \quad (185)$$

which concludes the proof of Theorem 6. ■

VIII. CONCLUDING REMARKS AND EXTENSIONS

In this paper, we proposed a new receiver architecture for MIMO channels that bridges the performance gap between traditional linear receivers and the optimal joint ML decoder. We studied the scenario without coding across the transmit antennas and found that the proposed integer-forcing linear receiver performs close to the optimal joint receiver while incurring only some additional complexity over traditional linear receivers. We characterized the diversity multiplexing tradeoff of the proposed architecture and showed that it is the same as that of the joint ML decoder. We considered a generalized MIMO channel model with interference and found that the integer-forcing receiver can be used to effectively mitigate interference in an oblivious. Furthermore, the proposed architecture achieves the same generalized degrees-of-freedom as the joint ML decoder. An interesting question for future work is how the integer-forcing architecture behaves when coding across transmit antennas is permitted.

APPENDIX A

INTEGER-FORCING VS. V-BLAST IV

Recall that V-BLAST II performs decoding in the optimal order and V-BLAST III allows for rate allocation. In this appendix, we introduce V-BLAST IV, which allows for both rate allocation and an optimized decoding order. Under V-BLAST IV, the data streams are decoded with respect to the ordering

$$\pi^* = \arg \max_{\pi \in \Pi} \min_m 2MR_{\pi(m)}(\mathbf{H}). \quad (186)$$

where $R_{\pi(m)}(\mathbf{H})$ is given by (121). We compare the behavior of V-BLAST IV to that of the integer-forcing linear receiver in Figures 13, 14, 15. The results show although V-BLAST IV achieves good performance for low to medium SNR, the integer-forcing linear achieves higher outage rates and lower outage probabilities in the medium to high SNR regime.

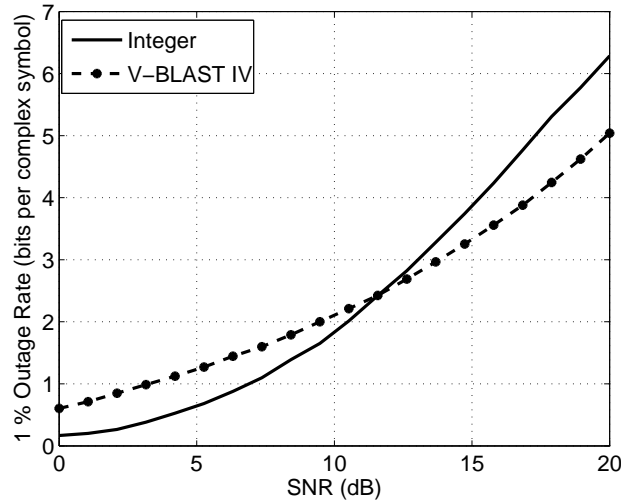


Fig. 13. 1 percent outage rates for the 2×2 complex-valued MIMO channel with Rayleigh fading.

APPENDIX B

PROOF OF THEOREM 4

In order to establish Theorem 4, we need a few key facts about lattices.

Definition 12 (Lattice): A lattice $\Lambda \subset \mathbb{R}^{2M}$ is a set of points that satisfy the following properties:

$$\text{i) } \mathbf{0} \in \Lambda \quad (187)$$

$$\text{ii) if } \mathbf{x}, \mathbf{y} \in \Lambda \text{ then } \mathbf{x} + \mathbf{y} \in \Lambda. \quad (188)$$

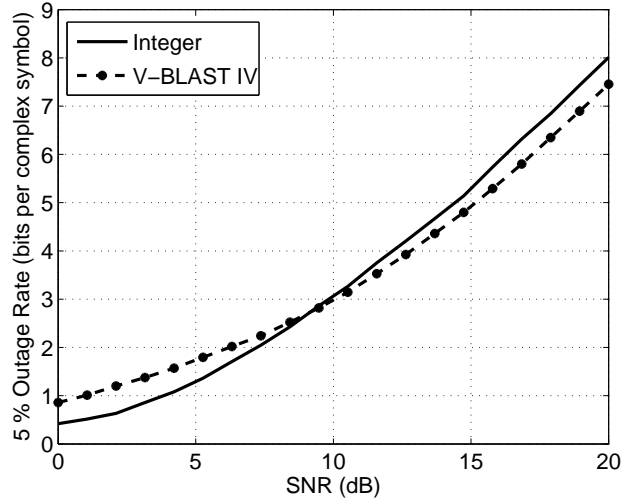


Fig. 14. 5 percent outage rates for the 2×2 complex-valued MIMO channel with Rayleigh fading.

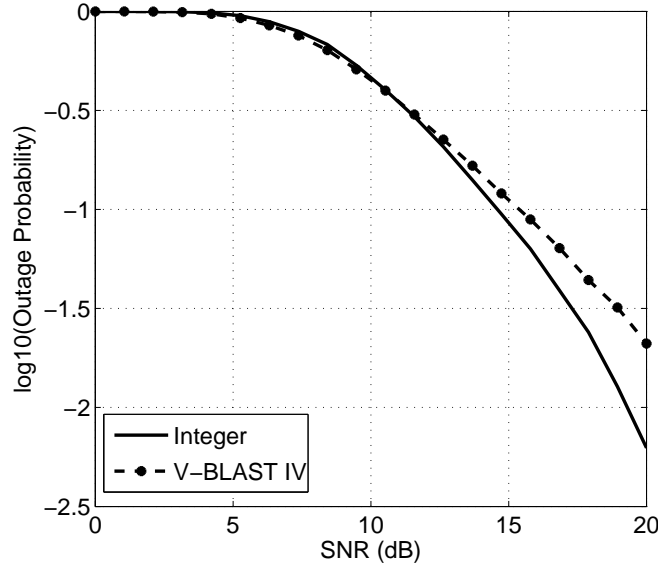


Fig. 15. Outage probability for the 2×2 complex-valued MIMO channel with Rayleigh fading for a target sum rate of $R = 6$.

We call the rank- L matrix \mathbf{G} a generator matrix for Λ if

$$\Lambda = \{ \mathbf{G}\mathbf{d} : \mathbf{d} \in \mathbb{Z}^{2M} \} \quad (189)$$

We use the definition of dual lattices from [35].

Definition 13 (Dual Lattice): Given a lattice $\Lambda \subset \mathbb{R}^{2M}$ with a rank- L generator matrix \mathbf{G} , the dual lattice Λ^* has generator matrix $(\mathbf{G}^T)^\dagger$,

$$\Lambda^* = \{ (\mathbf{G}^T)^\dagger \mathbf{d} : \mathbf{d} \in \mathbb{Z}^{2M} \} . \quad (190)$$

To prove Theorem 4, we consider *successive minima* for the involved lattices, a standard concept from the Diophantine approximation literature (see e.g. [36]–[38]), defined as follows.

Definition 14 (Successive Minima): Let $\mathcal{B} = \{ \mathbf{x} \in \mathbb{R}^{2M} : \|\mathbf{x}\| \leq 1 \}$ be the unit ball. Given a lattice $\Lambda \subset \mathbb{R}^{2M}$ with a rank- L generator matrix, the m^{th} successive minimum $\epsilon_m(\Lambda)$ where $1 \leq m \leq L$ is given by

$$\epsilon_m(\Lambda) = \{ \min \epsilon : \exists m \text{ linearly independent lattice points } \mathbf{v}_1, \dots, \mathbf{v}_m \in \Lambda \cap \epsilon \mathcal{B} \}$$

Remark 9: Definition 14 implies that $\epsilon_1(\Lambda) \leq \epsilon_2(\Lambda) \leq \dots \leq \epsilon_L(\Lambda)$ for any lattice Λ .

The following basic property linking the successive minima of a lattice with those of its dual lattice is key to our proof.

Lemma 3 ([35, Proposition 3.3]): Let $\Lambda \subset \mathbb{R}^{2M}$ be an arbitrary lattice with a rank- L generator matrix and Λ^* be its dual lattice. The successive minima for Λ and Λ^* satisfy the following inequality:

$$\epsilon_\ell^2(\Lambda^*)\epsilon_1^2(\Lambda) \leq \frac{m^2(m+3)}{4} \text{ for } m = 1, 2, \dots, L. \quad (191)$$

Finally, we also need the following result concerning a random Gaussian lattice.

Lemma 4 ([39, Lemma 3]): Let $\mathbf{H} \in \mathbb{R}^{2N \times 2M}$ be the real-valued decomposition of a $N \times M$ complex Gaussian matrix with i.i.d. Rayleigh entries. Let $\Lambda = \{\mathbf{H}\mathbf{d} : \mathbf{d} \in \mathbb{Z}^{2M}\}$ be the lattice generated by \mathbf{H} . Then

$$\Pr(\epsilon_1(\Lambda) \leq s) = \begin{cases} \gamma s^{2N}, & M < N, \\ \delta s^{2N} \max\{-(\ln s)^{N+1}, 1\}, & M = N. \end{cases}$$

where γ and δ are constants independent of s .

Proof: (Theorem 4). Let $R = r \log \text{SNR}$ be the target rate where $r \in [0, M]$. For a fixed set of equations $\mathbf{A} = [\mathbf{a}_1, \dots, \mathbf{a}_{2M}]^T$ and a fixed preprocessing matrix $\mathbf{B} = [\mathbf{b}_1, \dots, \mathbf{b}_{2M}]^T$, the outage probability is given by

$$\begin{aligned} p_{\text{OUT}}(r, \mathbf{A}, \mathbf{B}) &= \Pr\left(R(\mathbf{H}, \mathbf{A}, \mathbf{B}) < r \log \text{SNR}\right) \\ &= \Pr\left(\min_m R(\mathbf{H}, \mathbf{a}_m, \mathbf{b}_m) < \frac{r}{2M} \log \text{SNR}\right) \\ &= \Pr\left(\max_m \|\mathbf{b}_m\|^2 + \text{SNR} \|\mathbf{H}^T \mathbf{b}_m - \mathbf{a}_m\|^2 > \text{SNR}^{1-\frac{r}{M}}\right) \end{aligned}$$

For a fixed set of equations \mathbf{A} , we are free to choose any projection matrix \mathbf{B} , resulting in the following bound:

$$\begin{aligned} p_{\text{OUT}}(r, \mathbf{A}) &= \min_{\mathbf{B}} p_{\text{OUT}}(r, \mathbf{A}, \mathbf{B}) \\ &\leq p_{\text{OUT}}(r, \mathbf{A}, \mathbf{A}\mathbf{H}^\dagger) \\ &= \Pr\left(\max_m \left\|(\mathbf{H}^T)^\dagger \mathbf{a}_m\right\|^2 > \text{SNR}^{1-\frac{r}{M}}\right) \end{aligned}$$

We then choose the best set of full-rank equations by optimizing (192) over all integer matrices $\mathbf{A} \in \mathbb{Z}^{2M \times 2M}$ with non-zero determinant:

$$p_{\text{OUT}}(r) = \min_{\mathbf{A}: |\mathbf{A}| > 0} p_{\text{OUT}}(r, \mathbf{A}) \quad (192)$$

$$\leq \min_{\mathbf{A}: |\mathbf{A}| > 0} \Pr\left(\max_m \left\|(\mathbf{H}^T)^\dagger \mathbf{a}_m\right\|^2 > \text{SNR}^{1-\frac{r}{M}}\right) \quad (193)$$

$$= \Pr\left(\min_{\mathbf{A}: |\mathbf{A}| > 0} \max_m \left\|(\mathbf{H}^T)^\dagger \mathbf{a}_m\right\|^2 > \text{SNR}^{1-\frac{r}{M}}\right) \quad (194)$$

We use properties of dual lattices to bound (194). For a fixed \mathbf{H} , let Λ_{CHANNEL} be the lattice generated by \mathbf{H} and Λ_{DUAL} be the dual lattice generated by $(\mathbf{H}^T)^\dagger$,

$$\Lambda_{\text{CHANNEL}} = \{\mathbf{H}\mathbf{d} : \mathbf{d} \in \mathbb{Z}^{2M}\} \quad (195)$$

$$\Lambda_{\text{DUAL}} = \left\{(\mathbf{H}^T)^\dagger \mathbf{d} : \mathbf{d} \in \mathbb{Z}^{2M}\right\}. \quad (196)$$

Using the definition of successive minima (Definition 14), it follows that

$$\min_{\mathbf{A}: |\mathbf{A}| > 0} \max_m \left\|(\mathbf{H}^T)^\dagger \mathbf{a}_m\right\|^2 = \max_{m=1, \dots, 2M} \epsilon_m(\Lambda_{\text{DUAL}}) \quad (197)$$

$$= \epsilon_{2M}(\Lambda_{\text{DUAL}}). \quad (198)$$

We now express (194) in terms of the successive minima of Λ_{DUAL} ,

$$p_{\text{OUT}}(r) \leq \Pr \left(\min_{\mathbf{A}: |\mathbf{A}| > 0} \max_m \left\| (\mathbf{H}^T)^\dagger \mathbf{a}_m \right\|^2 > \text{SNR}^{1-\frac{r}{M}} \right) \quad (199)$$

$$= \Pr \left(\epsilon_{2M}^2(\Lambda_{\text{DUAL}}) > \text{SNR}^{1-\frac{r}{M}} \right) \quad (200)$$

Using Lemma 3, we can bound the successive minima of Λ_{DUAL} in terms of the successive minima of Λ_{CHANNEL} ,

$$\epsilon_{2M}^2(\Lambda_{\text{DUAL}}) \leq \frac{2M^3 + 3M^2}{\epsilon_1^2(\Lambda_{\text{CHANNEL}})} . \quad (201)$$

Combining (200) and (201), the outage probability is upper bounded by

$$p_{\text{OUT}}(r) \leq \Pr \left(\frac{2M^3 + 3M^2}{\epsilon_1^2(\Lambda_{\text{CHANNEL}})} > \text{SNR}^{1-\frac{r}{M}} \right) \quad (202)$$

$$= \Pr \left(\epsilon_1^2(\Lambda_{\text{CHANNEL}}) < \frac{2M^3 + 3M^2}{\text{SNR}^{1-\frac{r}{M}}} \right) \quad (203)$$

This probability can in turn be upper bounded using Lemma 4. For large SNR, we find that

$$p_{\text{OUT}}(r) \leq \frac{\max\{\gamma, \delta\} (2M^3 + 3M^2)^N (\ln \text{SNR})^{N+1}}{\text{SNR}^{N(1-\frac{r}{M})}} .$$

where γ, δ are constants independent of SNR. The achievable diversity for multiplexing gain r is thus

$$d(r) = \lim_{\text{SNR} \rightarrow \infty} \frac{-\log \rho_{\text{OUT}}(r)}{\text{SNR}} \quad (204)$$

$$\geq \lim_{\text{SNR} \rightarrow \infty} \frac{N(1 - \frac{r}{M}) \text{SNR}}{\text{SNR}} - \frac{o(\text{SNR})}{\text{SNR}} \quad (205)$$

$$= N \left(1 - \frac{r}{M} \right) \quad (206)$$

■

APPENDIX C PROOF OF LEMMA 2

In order to prove Lemma 2, we use a technique introduced in [40]. We first construct semi-norms $f : \mathbb{R}^{2M} \rightarrow \mathbb{R}$ and $g : \mathbb{R}^{2M} \rightarrow \mathbb{R}$ as well as a norm $h : \mathbb{R}^{2M} \rightarrow \mathbb{R}$. We then apply Minkowski's Theorem (see Theorem 7 below) to find $2M$ linearly independent integer vectors that achieve the successive minima (with respect to the norm h). Afterwards, we will show that these integer vectors satisfy the conditions in Lemma 2. We will need the following definitions and theorems in the proof.

Definition 15 (h -Unit Ball): Let $h : \mathbb{R}^{2M} \rightarrow \mathbb{R}$ be a norm. The unit ball with respect to h (or h -unit ball) is denoted by

$$\mathcal{B}_h = \{ \mathbf{x} \in \mathbb{R}^{2M} : h(\mathbf{x}) \leq 1 \} . \quad (207)$$

The volume of \mathcal{B}_h is denoted by V_h .

Definition 16 (Successive h -Minima): Let $h : \mathbb{R}^{2M} \rightarrow \mathbb{R}$ be a norm and \mathcal{B}_h be the h -unit ball. The m^{th} successive h -minimum ϵ_m is given by

$$\epsilon_m = \left\{ \min \epsilon : \exists m \text{ linearly independent integer points } \mathbf{v}_1, \dots, \mathbf{v}_m \in \mathbb{Z}^{2M} \cap \epsilon \mathcal{B}_h \right\}$$

for $1 \leq m \leq 2M$.

Theorem 7 (Minkowski): For any norm $h : \mathbb{R}^{2M} \rightarrow \mathbb{R}$, the successive h -minima satisfy

$$V_h \prod_{i=1}^{2M} \epsilon_i \leq 2^{2M} . \quad (208)$$

Theorem 8 (Dirichlet): For any $\mathbf{T} \in \mathbb{R}^{2K \times (2M-2K)}$ and $N \in \mathbb{Z}_+$, there exists a $[\mathbf{q}^T, \mathbf{p}^T]^T \in \mathbb{Z}^{2M-2K} \times \mathbb{Z}^{2K} \setminus \mathbf{0}$ such that

$$\|\mathbf{q}\|_\infty \leq Q \quad (209)$$

$$\|\mathbf{T}\mathbf{q} - \mathbf{p}\|_\infty \leq \frac{1}{Q^{\frac{M-K}{K}}} \quad (210)$$

Theorem 9 (Khinchin-Groshev): Fix a function $\Psi : \mathbb{N} \rightarrow \mathbb{R}^+$. If

$$\sum_{q=1}^{\infty} q^{2M-2K-1} (\Psi(q))^{2K} < \infty, \quad (211)$$

then, for almost all $\mathbf{T} \in \mathbb{R}^{2K \times (2M-2K)}$ satisfying $|t_{i,j}| \leq 1$ for all i, j , there are only finitely many solutions $[\mathbf{q}^T, \mathbf{p}^T]^T \in \mathbb{Z}^{2M-2K} \times \mathbb{Z}^{2K}$ to the inequality

$$\|\mathbf{T}\mathbf{q} - \mathbf{p}\|_\infty < \Psi(\|\mathbf{q}\|_\infty). \quad (212)$$

Theorem 7 can be found in [38, Equation 1.1]. Theorems 8 and 9 can be found in many standard Diophantine approximation texts (see, for instance, [36]).

Proof: (Lemma 2). For any vector $\mathbf{v} \in \mathbb{Z}^{2M}$, we denote the first $2M-2K$ components by \mathbf{q} and the remaining $2K$ components by \mathbf{p} , and will thus write

$$\mathbf{v} = \begin{bmatrix} \mathbf{q} \\ \mathbf{p} \end{bmatrix}. \quad (213)$$

From the lemma statement, $\mathbf{T} = [\mathbf{t}_1 \cdots \mathbf{t}_{2K}]^T$ is a (rationally independent) $2K \times (2M-2K)$ real-valued matrix with $|t_{i,j}| \leq 1$ for all i, j . For a fixed \mathbf{T} , we define the semi-norms f, g as follows:

$$f(\mathbf{v}) = \|\mathbf{T}\mathbf{q} - \mathbf{p}\| \quad (214)$$

$$g(\mathbf{v}) = \|\mathbf{q}\|. \quad (215)$$

For a fixed $Q \geq 2M$, we let λ_1 denote the minimum value of $f(\mathbf{v})$ under the constraint $\|\mathbf{q}\| \leq Q$,

$$\lambda_1 = \min_{\substack{\mathbf{v} \in \mathbb{Z}^{2M} \setminus \{\mathbf{0}\} \\ g(\mathbf{v}) \leq Q}} f(\mathbf{v}) \quad (216)$$

$$= \min_{\substack{\mathbf{q} \in \mathbb{Z}^{2M-2K} \\ \|\mathbf{q}\| \leq Q}} \min_{\substack{\mathbf{p} \in \mathbb{Z}^{2K} \\ [\mathbf{q}^T, \mathbf{p}^T]^T \neq \mathbf{0}}} \|\mathbf{T}\mathbf{q} - \mathbf{p}\|, \quad (217)$$

$\mathbf{q}_1 \in \mathbb{Z}^{2M}$ denote the integer vector that achieves λ_1

$$\mathbf{q}_1 = \arg \min_{\substack{\mathbf{q} \in \mathbb{Z}^{2M-2K} \\ \|\mathbf{q}\| \leq Q}} \min_{\substack{\mathbf{p} \in \mathbb{Z}^{2K} \\ [\mathbf{q}^T, \mathbf{p}^T]^T \neq \mathbf{0}}} \|\mathbf{T}\mathbf{q} - \mathbf{p}\|, \quad (218)$$

and μ_1 be the value of \mathbf{q}_1 evaluated by g ,

$$\mu_1 = g(\mathbf{q}_1) \quad (219)$$

$$= \|\mathbf{q}_1\|. \quad (220)$$

Based on the seminorms f and g , we define the function $h : \mathbb{R}^{2M} \rightarrow \mathbb{R}$ as follows:

$$h(\mathbf{v}) = \left(f^2(\mathbf{v}) + \frac{\lambda_1^2}{\mu_1^2} g^2(\mathbf{v}) \right)^{1/2} \quad (221)$$

$$= \left(\|\mathbf{T}\mathbf{q} - \mathbf{p}\|^2 + \frac{\lambda_1^2}{\mu_1^2} \|\mathbf{q}\|^2 \right)^{1/2}. \quad (222)$$

In the sequel, we show that h is a norm for $Q > 1$. We define the $2M \times 2M$ matrix $\mathbf{\Gamma}$ as follows:

$$\mathbf{\Gamma} = \begin{bmatrix} \mathbf{T} & -\mathbf{I}_{2K} \\ \frac{\lambda_1}{\mu_1} \mathbf{I}_{2M-2K} & \mathbf{0} \end{bmatrix}$$

Note that we can rewrite the function h using $\mathbf{\Gamma}$:

$$h(\mathbf{v}) = \|\mathbf{\Gamma}\mathbf{v}\|. \quad (223)$$

Since exchanging rows of a matrix only affects the sign of its determinant, we have that

$$|\det(\mathbf{\Gamma})| = \left| \det \left(\begin{bmatrix} \frac{\lambda_1}{\mu_1} \mathbf{I}_{2M-2K} & \mathbf{0} \\ \mathbf{T} & -\mathbf{I}_{2K} \end{bmatrix} \right) \right|. \quad (224)$$

Now we use the fact that the determinant of a lower triangular matrix is just the product of its diagonal entries,

$$|\det(\mathbf{\Gamma})| = \left(\frac{\lambda_1}{\mu_1} \right)^{2M-2K}. \quad (225)$$

Consider the case where $Q > 1$. Since \mathbf{T} is rationally independent, it follows that $\lambda_1 > 0$. Since $\mu_1 \geq 0$ by definition, we have that $\frac{\lambda_1}{\mu_1} > 0$. Since $\mathbf{\Gamma}$ is full-rank and thus injective, h is a norm.

Let V_h be the volume of the h -unit ball. Let $\mathbf{u} = \mathbf{T}\mathbf{v}$. It follows that:

$$V_h = \int_{\{\mathbf{v}: \|\mathbf{\Gamma}\mathbf{v}\| \leq 1\}} d\mathbf{v} \quad (226)$$

$$= \int_{\{\mathbf{u}: \|\mathbf{u}\| \leq 1\}} |\det(\mathbf{\Gamma}^{-1})| d\mathbf{u} \quad (227)$$

$$= \frac{1}{|\det(\mathbf{\Gamma})|} \int_{\{\mathbf{u}: \|\mathbf{u}\| \leq 1\}} d\mathbf{u} \quad (228)$$

$$= \frac{1}{|\det(\mathbf{\Gamma})|} V_h \quad (229)$$

$$= \left(\frac{\mu_1}{\lambda_1} \right)^{2M-2K} V_{2M}, \quad (230)$$

where V_{2M} is the volume of the unit ball with respect to the Euclidean norm (in $2M$ dimensional space).

Let $\epsilon_1, \dots, \epsilon_{2M}$ be the successive minima with respect to h (see Definition 16). Let $\mathbf{y}_1, \dots, \mathbf{y}_{2M} \in \mathbb{Z}^{2M}$ be the linearly independent integer points that achieve the successive minima, i.e:

$$h(\mathbf{y}_i) = \epsilon_i. \quad (231)$$

Using Minkowski's Theorem on successive minima (Theorem 7), we have that:

$$V_h \prod_{i=1}^{2M} \epsilon_i \leq 2^{2M}, \quad (232)$$

where V_h is the volume of the h -unit ball. Using (230), we have that:

$$\left(\frac{\mu_1}{\lambda_1} \right)^{2M-2K} V_{2M} \prod_{i=1}^{2M} \epsilon_i \leq 2^{2M}. \quad (233)$$

Rewriting the above, we have that

$$\left(\frac{\mu_1}{\lambda_1} \right)^{2M-2K} \prod_{i=1}^{2M} \epsilon_i \leq C, \quad (234)$$

where C is a constant that depends only on $2M$. Rearranging (234), we arrive at the following:

$$\epsilon_{2M} \leq C \left(\frac{\lambda_1}{\epsilon_1} \cdots \frac{\lambda_1}{\epsilon_{2M-2K}} \right) \left(\frac{1}{\epsilon_{2M-2K+1} \cdots \epsilon_{2M-1}} \right) \left(\frac{1}{\mu_1^{2M-2K}} \right) \quad (235)$$

$$= C \left(\frac{\lambda_1}{\epsilon_1} \cdots \frac{\lambda_1}{\epsilon_{2M-2K}} \right) \left(\frac{\lambda_1}{\epsilon_{2M-2K+1}} \cdots \frac{\lambda_1}{\epsilon_{2M-1}} \right) \left(\frac{1}{\mu_1^{2M-2K} \lambda_1^{2K-1}} \right) \quad (236)$$

$$= C \left(\frac{\lambda_1}{\epsilon_1} \cdots \frac{\lambda_1}{\epsilon_{2M-1}} \right) \left(\frac{1}{\mu_1^{2M-2K} \lambda_1^{2K-1}} \right) \quad (237)$$

$$= C \left(\frac{\lambda_1}{\epsilon_1} \cdots \frac{\lambda_1}{\epsilon_{2M-1}} \right) \left(\frac{\lambda_1}{\mu_1^{2M-2K} \lambda_1^{2K}} \right). \quad (238)$$

For all $\mathbf{v} \in \mathbb{Z}^{2M} \setminus \{0\}$, we have that $h(\mathbf{v}) \geq \lambda_1$. To see this, we can consider the case where $\|\mathbf{q}\| < \mu_1$ and $\|\mathbf{q}\| \geq \mu_1$ separately. When $\|\mathbf{q}\| \geq \mu_1$, h can be bounded as follows

$$h(\mathbf{v}) = \left(\|\mathbf{T}\mathbf{q} - \mathbf{p}\|^2 + \frac{\lambda_1^2}{\mu_1^2} \|\mathbf{q}\|^2 \right)^{1/2} \quad (239)$$

$$\geq \frac{\lambda_1}{\mu_1} \|\mathbf{q}\| \quad (240)$$

$$\geq \lambda_1. \quad (241)$$

We now consider the case where $\|\mathbf{q}\| < \mu_1$. We first bound h as follows

$$h(\mathbf{v}) = \left(\|\mathbf{T}\mathbf{q} - \mathbf{p}\|^2 + \frac{\lambda_1^2}{\mu_1^2} \|\mathbf{q}\|^2 \right)^{1/2} \quad (242)$$

$$\geq \|\mathbf{T}\mathbf{q} - \mathbf{p}\|. \quad (243)$$

Recall that $\mu_1 = \|\mathbf{q}_1\|$ and $\lambda_1 = \min_{\mathbf{p} \in \mathbb{Z}^{2K}} \|\mathbf{T}\mathbf{q}_1 - \mathbf{p}\|$. Assume that there exists a \mathbf{q} with $\|\mathbf{q}\| < \|\mathbf{q}_1\|$ such that

$$\min_{\mathbf{p}} \|\mathbf{T}\mathbf{q} - \mathbf{p}\| < \min_{\mathbf{p}} \|\mathbf{T}\mathbf{q}_1 - \mathbf{p}\| \quad (244)$$

$$= \lambda_1, \quad (245)$$

then the definition of \mathbf{q}_1 in (218) is violated. Hence, in this case, $h(\mathbf{v}) \geq \|\mathbf{T}\mathbf{q} - \mathbf{p}\| \geq \lambda_1$.

Since $\epsilon_j = h(\mathbf{y}_j)$ for some $\mathbf{y}_j \in \mathbb{R}^{2M}$, it follows that

$$\epsilon_j \geq \lambda_1 \quad \text{for all } j = 1, \dots, 2M. \quad (246)$$

Combing the above with (238), it follows that

$$\epsilon_{2M} \leq C \frac{\lambda_1}{\mu_1^{2M-2K} \lambda_1^{2K}}. \quad (247)$$

From the definition of h , ϵ_{2M} , and $\mathbf{y}_1 \cdots \mathbf{y}_{2M}$, we have that

$$h(\mathbf{y}_j) \leq \epsilon_{2M} \quad \text{for } j = 1 \cdots 2M. \quad (248)$$

By the construction of h (see (221)), we have that:

$$f(\mathbf{y}_j) \leq h(\mathbf{y}_j) \leq \epsilon_{2M} \leq C \frac{\lambda_1}{\lambda_1^{2K} \mu_1^{2M-2K}} \quad (249)$$

$$\frac{\lambda_1}{\mu_1} g(\mathbf{y}_j) \leq h(\mathbf{y}_j) \leq \epsilon_{2M} \leq C \frac{\lambda_1}{\lambda_1^{2K} \mu_1^{2M-2K}}, \quad (250)$$

for $j = 1, \dots, 2M$. The above equations imply that

$$f(\mathbf{y}_j) \leq C \frac{\lambda_1}{\lambda_1^{2K} \mu_1^{2M-2K}} \quad (251)$$

$$g(\mathbf{y}_j) \leq C \frac{\mu_1}{\lambda_1^{2K} \mu_1^{2M-2K}}, \quad (252)$$

for $j = 1, \dots, 2M$.

Recall that λ_1, μ_1 are defined with respect to a fixed Q . We now show that for sufficiently large Q ,

$$\lambda_1(Q)^{2K} \mu_1(Q)^{2M-2K} \geq \frac{1}{\log(\mu_1(Q))^2}. \quad (253)$$

We define the function $\Psi : \mathbb{Z} \rightarrow \mathbb{R}$ as follows

$$\Psi(q) = 1 \quad \text{for } q = 1 \quad (254)$$

$$\Psi(q) = \frac{1}{q^{\frac{2M-2K}{2K}} (\log q)^{\frac{2}{2K}}} \quad \text{for } q > 1. \quad (255)$$

We note that with this choice of Ψ , it follows that

$$\sum_q q^{2M-2K-1} \Psi(q)^{2K} < \infty. \quad (256)$$

Applying Theorem 9, we have that for rationally independent $\mathbf{T} \in \mathbb{R}^{2K \times (2M-2K)}$, there are only a finite number of integer solutions $[\mathbf{q}^T, \mathbf{p}^T]^T \in \mathbb{Z}^{2M-2K} \times \mathbb{Z}^{2K}$ that satisfy the following condition:

$$\|\mathbf{T}\mathbf{q} - \mathbf{p}\|_\infty < \frac{1}{\|\mathbf{q}\|_\infty^{\frac{2M-2K}{2K}} (\log \|\mathbf{q}\|_\infty)^{\frac{2}{2K}}}. \quad (257)$$

We rewrite this condition as follows

$$\|\mathbf{q}\|_\infty^{2M-2K} \|\mathbf{T}\mathbf{q} - \mathbf{p}\|_\infty^{2K} < \frac{1}{(\log \|\mathbf{q}\|_\infty)^2}. \quad (258)$$

We first fix an rationally independent $\mathbf{T} \in \mathbb{R}^{2K \times (2M-2K)}$. Recall from (218) that $\mathbf{q}_1(Q)$ is the integer vector that achieves $\lambda_1(Q)$ for a given Q . Clearly, $\{\|\mathbf{q}_1(Q)\|_\infty\}_Q$ is a non-decreasing integer sequence (in Q). Assume that $\|\mathbf{q}_1(Q)\|_\infty$ is unbounded as $Q \rightarrow \infty$. By Theorem 9, we know that there are only a finite number of integers \mathbf{q} that satisfy the condition in (258). Let \mathbf{q}' be the integer with the largest L_∞ norm that satisfies the condition in (258). This suggests that for all Q where $\|\mathbf{q}_1(Q)\|_\infty > \|\mathbf{q}'\|_\infty$, $\mathbf{q}_1(Q)$ does not satisfy the condition in (258). Since $\{\|\mathbf{q}_1(Q)\|_\infty\}_Q$ is an unbounded non decreasing sequence, there exists some Q' such that for all $Q > Q'$, $\mathbf{q}_1(Q)$ does not satisfy the condition in (258). Note that (253) follows since any \mathbf{q}, \mathbf{p} that satisfies

$$\|\mathbf{q}\|_\infty^{2M-2K} \|\mathbf{T}\mathbf{q} - \mathbf{p}\|_\infty^{2K} \geq \frac{1}{(\log \|\mathbf{q}\|_\infty)^2} \quad (259)$$

also satisfies

$$\|\mathbf{q}\|_\infty^{2M-2K} \|\mathbf{T}\mathbf{q} - \mathbf{p}\|_\infty^{2K} \geq \frac{1}{(\log \|\mathbf{q}\|_\infty)^2}. \quad (260)$$

Finally, for any rationally independent $\mathbf{T} \in \mathbb{R}^{2K \times (2M-2K)}$, we prove that sequence $\{\|\mathbf{q}_1(Q)\|_\infty\}_Q$ is unbounded as $Q \rightarrow \infty$. We prove this by contradiction. That is, assume that there exists some $C \in \mathbb{Z}_+$ such that $\|\mathbf{q}_1(Q)\|_\infty \leq C$ for all Q . This implies that $\mathbf{q}_1(Q)$ takes only a finite set of values. Hence, there exists a C' such that

$$\min_{\mathbf{p} \in \mathbb{Z}^{2K}} \|\mathbf{T}\mathbf{q}_1(Q) - \mathbf{p}\|_\infty \geq C' \quad (261)$$

for all Q . However, by definition of $\lambda_1(Q)$ and Dirichlet's theorem (Theorem 8) we have that

$$\min_{\mathbf{p} \in \mathbb{Z}^{2K}} \|\mathbf{T}\mathbf{q}_1(Q) - \mathbf{p}\|_\infty \leq \frac{1}{Q^{\frac{2M-2K}{2K}}} \quad (262)$$

for all $Q \in \mathbb{N}$. This results in a contradiction with our assumption. Dirichlet's Theorem (Theorem 8) is defined in terms of the ℓ_∞ norm and λ_1 is defined in terms of the ℓ_2 norm. Using the fact that

$$\lambda_1 = \min_{\substack{\mathbf{q} \in \mathbb{Z}^{2M-2K} \\ \|\mathbf{q}\| \leq Q}} \min_{\substack{\mathbf{p} \in \mathbb{Z}^{2K} \\ [\mathbf{q}^T, \mathbf{p}^T]^T \neq \mathbf{0}}} \|\mathbf{T}\mathbf{q} - \mathbf{p}\| \quad (263)$$

$$\leq \sqrt{2K} \min_{\substack{\mathbf{q} \in \mathbb{Z}^{2M-2K} \\ \|\mathbf{q}\| \leq Q}} \min_{\substack{\mathbf{p} \in \mathbb{Z}^{2K} \\ [\mathbf{q}^T, \mathbf{p}^T]^T \neq \mathbf{0}}} \|\mathbf{T}\mathbf{q} - \mathbf{p}\|_\infty, \quad (264)$$

and (262), we have that

$$\lambda_1 \leq \frac{\sqrt{2K}}{Q^{\frac{2M-2K}{2K}}}. \quad (265)$$

By definition, we have that

$$\mu_1 \leq Q. \quad (266)$$

Using (251), (253), (265), and (266), and assuming that Q is sufficiently large, we bound $f(\mathbf{y}_j)$ as follows:

$$f(\mathbf{y}_j) \leq C \frac{\lambda_1}{\lambda_1^{2K} \mu_1^{2M-2K}} \quad (267)$$

$$\leq C \lambda_1 (\log \mu_1)^2 \quad (268)$$

$$\leq C' \frac{(\log \mu_1)^2}{Q^{\frac{2M-2K}{2K}}} \quad (269)$$

$$\leq C' \frac{(\log Q)^2}{Q^{\frac{2M-2K}{2K}}}, \quad (270)$$

where C' is a constant that does not depend on Q . Similarly, we can bound $g(\mathbf{y}_j)$ as follows:

$$g(\mathbf{y}_j) \leq C \mu_1 (\log \mu_1)^2 \leq C' Q (\log Q)^2, \quad (271)$$

which concludes the proof of Lemma 2. ■

ACKNOWLEDGMENT

The authors are grateful to Or Ordentlich for pointing out the suboptimality of unimodular integer forcing, Giuseppe Caire for corrections in the proof of Theorem 6, and Seyong Park for many helpful discussions.

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